Signal norm testing in additive and independant standard Gaussian noise

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Abstract

This paper addresses signal norm testing (SNT), that is, the problem of deciding whether a random signal norm exceeds some specified value $\tau \geqslant 0$ or not, when the signal has unknown probability distribution and is observed in additive and independent standard Gaussian noise. The theoretical framework proposed for SNT extends usual notions in statistical inference and introduces a new optimality criterion. This one takes the invariance of both the problem and the noise distribution into account, via conditional notions of power and size and, more specifically, the introduction of the spherically-conditioned power function. The theoretical results established with respect to this criterion extend those deriving from standard statistical inference theory in the case of an unknown deterministic signal.

Thinkable applications are problems where signal amplitude deviations from some nominal reference must be detected above a certain tolerance τ, possibly chosen by the user on the basis of his experience and know-how. In this respect, the theoretical results of this paper are applied to an SNT formulation for the problem of detecting random signals in noise, with a specific focus on the case where the noise standard deviation is unknown.

Keywords

Signal norm testing, hypothesis testing, invariance, conditional power function, spherically-conditioned power function, invariant tests, tests with uniformly best invariant spherically-conditioned power (UBISCP)

Contents

1 Introduction

A basic problem in statistical signal processing is the detection of the presence or the absence of some signal in additive noise, on the basis of some measurement or observation. In many cases, the observation, the signal and noise are d-dimensional real vectors. If the signal is absent, the observation consists of noise only. If the signal is present, the observation is the sum of this signal and noise.

The signal is often assumed to be some unknown deterministic d-dimensional real vector θ. The problem of detecting θ in noise is then stated as the (non-Bayesian) hypothesis testing problem of accepting or rejecting the hypothesis $\theta = 0$ with a specified value for the false alarm probability, that is, the probability of falsely rejecting $\theta = 0$. Such a standard framework is questionable with regard to physics. To begin with, the signal deterministic model is an oversimplification of the reality and a random model should generally be preferred. In any case, in many applications, the signal depends on pairs of physical parameters, such as velocities and positions, that cannot simultaneously be known to arbitrary precision because of Heisenberg's uncertainty principle. In addition, even in the case where the signal is known to be 0 for some nominal values of its parameters, more or less big fluctuations around these nominal values can occur — due to environmental conditions for instance — and induce deviations of $\|\theta\|$ around 0, where $\|\cdot\|$ throughout stands for the standard euclidean norm in the space $\mathbb{R}^d$ of all the d-dimensional real vectors. Depending on the application, the detection of small deviations of $\|\theta\|$ around 0 can be of poor interest for the user and only relatively big ones must actually be detected. Therefore, whether the signal is assumed to be deterministic or random, testing the signal norm with respect to 0 may sometimes be too severe, and even paradoxical, because of unavoidable imprecision due to physics in the parameter setting. Thence, the idea to introduce some tolerance in the detection problem statement, this tolerance being possibly specified by the user himself on the basis of his experience and know-how with respect to a given environment or context.

With respect to the foregoing, the scope of the present paper is then signal norm testing (SNT) with respect to some non-negative real value τ, that is, the problem of testing whether the norm of a d-dimensional real random signal with unknown distribution exceeds τ or not, when this signal is observed in independent noise. The value τ is then called the tolerance of the SNT problem. The standard detection problem evoked at the beginning of this introduction is then the particular SNT problem with null tolerance. To the best of our knowledge, SNT is addressed here for the first time. In what follows, noise is assumed to be standard Gaussian, in the sense that it is centred, Gaussian distributed, with covariance matrix proportional to the $d \times d$ identity matrix $\mathbf{I}_d$. Following standard terminology, the signal will be said to be observed in independent and additive white Gaussian noise (AWGN). This assumption is acceptable in many cases of practical interest. The novelty brought by this paper is then threefold.

1) We introduce the SNT problem, whose applications are seemingly numerous. To treat this problem, an original theoretical framework is established to perform SNT of any random signal, with any unknown distribution, in independent AWGN. SNT of a random signal thus concerns a random event, in contrast to standard statistical inference aimed at testing an hypothesis on a parameter of a distribution family.

2) Many hypothesis testing problems considered in the literature concern unknown deterministic parameter vectors and exhibit invariance properties with respect to nuisance parameters. The invariance principle [1–3] is a particularly suitable statistical tool in such cases. In contrast to standard theory based on the invariance principle, the most general results established below in SNT apply to any random vector, whatever its distribution. In fact, to establish these results where the signal plays the role of a random parameter with unknown distribution, the standard invariance principle dedicated to the deterministic case does not apply and an alternative approach to deal with the natural invariance of both the SNT problem and the noise distribution is thus proposed. Since the SNT problem is not scale invariant, save for the null tolerance case, our general results are established under the assumption that the noise standard deviation is known and embrace all possible tolerance values.

3) Application of the invariance principle to the detection of an unknown deterministic signal in AWGN has received much attention to design tests invariant to nuisance parameters and, thus, robust to various contexts and applications. For instance, [4–8] address the case of a noise covariance matrix with known form, whereas [9–19] expose adaptive solutions for the case where the noise covariance matrix is unknown and secondary data are available. These solutions takes the scale-invariance of the detection problem into account. As already mentioned above, the detection of a signal in AWGN, whatever its distribution, is the particular SNT problem with null tolerance. This paper then contributes also to the standard detection problem by presenting results that apply to any random signal with any unknown distribution, whether the noise standard deviation is known or estimated via a noise reference.

Application of SNT is thinkable any time a deviation from a nominal reference must be detected. For instance, beyond the standard detection problem, the results stated below could be helpful in tracking tasks where a tolerance may help select or pre-classify targets, in combination with a scheduling policy aimed at deciding which and how long certain targets must be tracked with high priority [20]. Similarly, for anti-collision radars or a robot asked to find its path in a certain environment, SNT could apply to the detection (resp. the deletion) of new (resp. old) obstacles, as well as the management of "tracked obstacles in a thresholded proximity of measurement" [21]. Fault-detection and structural health monitoring (SHM) could also be natural applications of what follows. "Because the stress level in any element will never be exactly zero, one must establish a threshold stress level for proper damage diagnosis" [22]. The introduction of a tolerance, aimed at bracketing possible fluctuations other than noise around the signal nominal model, could therefore be considered. Fault-detection, robust to system uncertainties and external noise, is still a challenging task addressed in most recent papers [23–25] and could possibly benefit from the theoretical SNT framework established in the sequel.

2 Outline of main results

Although our more general results concern SNT of a random signal with any unknown distribution, we begin with the case of an unknown deterministic signal. Albeit questionable with regard to physics for reasons evoked above, the deterministic case gives the opportunity to recall basics in statistical inference and establish some first easy results in SNT. These basic notions and results are then extended to deal with random signals and state our more generals theorems, which will then be applied to the detection problem. More specifically, three types of results are hereafter presented.

1) First, when the signal is assumed to be an unknown deterministic d-dimensional real vector, we tackle SNT within the usual framework of statistical inference. For the one-dimensional case, we can even exhibit uniformly most powerful (UMP) and uniformly most powerful unbiased (UMPU) tests, depending on the type of testing under consideration. In the general d-dimensional case, testing the signal norm with respect to τ at specified level $\gamma \in (0, 1)$ can be treated via the statistical invariance principle [1–4] to take the natural invariance of the problem into account and derive tests that are uniformly most powerful invariant (UMPI) with respect to the orthogonal group $\mathbf{O}_d$ in $\mathbb{R}^d$. Our main result in the deterministic case — namely, theorem 1 — then states that these UMPI tests are, in fact, UMP among the tests with level γ and spherically invariant power function. As such, they are said to be UMP-SIP. Theorem 1 is connected to Wald's theory of tests with uniformly best constant power (UBCP) [26]. The proof of theorem 1 will not be given in the section concerning the deterministic case because it is a straightforward consequence of our results established for random signals.

2) When the signal is random, which is a suitable model in many signal processing applications of practical interest, the decision-making in SNT concerns a random event, in contrast to standard statistical inference aimed at testing an hypothesis on an unknown parameter parameterizing a distribution family. Therefore, SNT of a random signal cannot be tackled via usual hypothesis testing. In addition, we make no assumption about the signal distribution. As a consequence, the natural invariance of the problem cannot be treated by means of the standard invariance principle because this one applies to problems involving distributions depending on parameters that are indeed unknown, but deterministic. Therefore, the problem of testing the norm of a random signal must be posed within an appropriate and dedicated mathematical framework. New definitions, extending those recalled in the deterministic case, are then introduced. A new criterion, suitable for the random case and based on the spherical invariance of both the problem and the Gaussian distribution, is proposed in coherence with these definitions. This criterion extends that of the deterministic case. The tests optimal with respect to this criterion are said to have uniformly best invariant spherically-conditioned power (UBISCP). Our main theoretical and most general results are then theorems 2 and 3. The former states that UBISCP tests are necessarily UMP-SIP and the latter that the UMP-SIP tests of theorem 1 are UBISCP, which extends their properties. The reader can already deduce from what precedes that theorem 1 actually follows from theorems 2 and 3.

3) The standard problem of detecting a random signal in independent AWGN is posed as an SNT problem. Theorem 3 applies and the performance measurements of UBISCP tests designed for various tolerances are discussed. The signal detection problem in case of an estimated noise standard deviation is considered as well because it can also be regarded as an SNT problem. In case of an unknown noise standard deviation, our results in SNT are adapted to detect random signals with any unknown distributions, via an *estimate-and-plug-in detector* [27] based on auxiliary data of noise alone. It then turns out that the use of a positive tolerance partly compensates the performance loss incurred by the use of the noise standard deviation estimate.

The next section introduces some material used throughout. SNT in the deterministic case is then treated in section 4 where the definitions of UMP, UMPU et UMPI tests are recalled since the sequel will often refer to such standard notions. Section 5 focuses on the general random case and leads to our main results. The application to signal detection is addressed in section 6 where the case of a known standard deviation and that of an unknown standard deviation are considered. Pespectives are then summarized in the concluding section 7 of this paper. For clarity sake, many mathematical proofs are postponed to appendices and only those that favour the understanding of the approach are kept in the main core of the paper.

3 Preliminary material

In this section, we present some notation and terminology as well as a few definitions that will be used throughout the rest of the paper with always the same meaning. We also state some preliminary results that will prove very useful and whose proofs are postponed to appendices.

To begin with, the tolerance with respect to which SNT is performed will always be denoted by τ. The corrupting noise will hereafter be denoted by X. It is assumed to be d-dimensional, centred, Gaussian distributed with covariance matrix $\mathbf{I}_d$. As usual, we write $X \sim \mathcal{N}(0, \mathbf{I}_d)$.

For any given $\rho \in [0,\infty)$, $\mathcal{R}(\rho,\cdot)$ hereafter stands for the cumulative distribution function of the square root of any random variable that follows the non-central χ^2 distribution with d degrees of freedom and non-central parameter ρ^2. Therefore, $\mathcal{R}$ is the map of $[0,\infty) \times [0,\infty)$ into $[0,1]$ such that, for any $\theta \in \mathbb{R}^d$ and any $\eta \in [0,\infty)$,

$$\mathrm{P}\left[\, \|\theta + X\| \in [0,\eta]\!] \,\right] = \mathcal{R}(\|\theta\|, \eta). \tag{1}$$

where $[0,\eta]\!]$ is any of the two intervals $[0,\eta]$ or $[0,\eta)$. Whether $[0,\eta]\!]$ is closed or not does not matter in the equality above because the probability distribution of $\theta + X$ is absolutely continuous with respect to Lebesgue's measure in $\mathbb{R}^d$. Given any $\rho \in [0,\infty)$, $\mathcal{R}(\rho,\cdot)$ is strictly increasing and continuous and, thus, a one-to-one mapping of $[0,\infty)$ into $[0,1)$. An analytical expression of $\mathcal{R}$ will be given in appendix IV for further technical use. In the main core of this paper, the definition of $\mathcal{R}$ given by (1) above suffices. The following lemmas state properties of $\mathcal{R}$ that will be very useful in the sequel.

Lemma 1 Given any $\eta \in (0,\infty)$, the map $\mathcal{R}(\cdot,\eta)$ is strictly decreasing.

PROOF: See appendix I. ∎

Lemma 2

(i) given $\gamma \in (0,1]$ and $\rho \in [0,\infty)$, there exists a unique solution $\lambda_\gamma(\rho) \in [0,\infty)$ in η to the equation $1 - \mathcal{R}(\rho,\eta) = \gamma$;

(ii) given $\gamma \in (0,1]$, λ_γ is a strictly increasing and everywhere continuous map of $[0,\infty)$ into $[0,\infty)$;

(iii) given $\rho \in [0,\infty)$, the map $\gamma \in (0,1] \mapsto \lambda_\gamma(\rho) \in [0,\infty)$ is strictly decreasing and continuous everywhere.

PROOF: See appendix II. ∎

In the sequel, a *test* is any measurable map of $\mathbb{R}^d$ into $\{0,1\}$. As usual, a test $\mathcal{T}$ is said to accept (resp. reject) a given hypothesis whenever it takes the value 0 (resp. the value 1). Thresholding tests, which play a crucial role in the sequel, are defined as follows. Given $\eta \in [0,\infty)$, a thresholding test with threshold height η is any test $\mathcal{T}_\eta$ such that

$$\mathcal{T}_\eta(y) = \begin{cases} 0 & \text{if} \quad \|y\| < \eta \\ 1 & \text{if} \quad \|y\| > \eta \end{cases} \tag{2}$$

or such that

$$\mathcal{T}_\eta(y) = \begin{cases} 0 & \text{if} \quad \|y\| > \eta \\ 1 & \text{if} \quad \|y\| < \eta. \end{cases} \tag{3}$$

If a thresholding test $\mathcal{T}_\eta$ with threshold η satisfies (2) (resp. (3)), it is said to be from above (resp. from below). The handling of equality in the definition of a thresholding test plays no role in what follows because of the absolute continuity of the observation probability distribution with respect to Lebesgue's measure.

Given any test $\mathcal{T}$, the *power function* of $\mathcal{T}$, with respect to family $\{\mathcal{N}(\theta,\mathbf{I}_d) : \theta \in \mathbb{R}^d\}$ of distributions, is defined for every $\theta \in \mathbb{R}^d$ by [1]

$$\beta_\theta(\mathcal{T}) = \mathrm{P}\left[\,\mathcal{T}(\theta + X) = 1\,\right]. \tag{4}$$

The value $\beta_\theta(\mathcal{T})$ is thus the probability that $\mathcal{T}$ rejects the hypothesis, whatever this hypothesis may be, when $Y = \theta + X$. In the sequel, we simply speak of the power function of test $\mathcal{T}$, without recalling the family distribution with respect to which it is defined since this family will remain the same.

Because of the spherical invariance of both the noise distribution and the testing problems encountered below, spheres of $\mathbb{R}^d$ will play an important role. For any given $\rho \in [0,\infty)$, the standard notation $\rho\mathbb{S}^{d-1}$ will hereafter stand for the sphere centred at the origin in $\mathbb{R}^d$ with radius ρ.

4 Signal norm testing in the deterministic case

This section can be regarded as an introduction to the more general random case. In particular, it gives the opportunity to recall basic definitions in statistical inference that will be extended in the random case. It also pinpoints the importance of thresholding tests due to the invariance of the SNT problem.

4.1 Problem statement

The observation Y is assumed to be Gaussian distributed with covariance matrix $\mathbf{I}_d$ and unknown mean $\theta \in \mathbb{R}^d$. As usual, we write $Y \sim \mathcal{N}(\theta, \mathbf{I}_d)$. The basic purpose of SNT in the deterministic case is then to decide whether $\|\theta\|$ is above some given real number τ or not. There are actually four hypotheses that can be tested: $\|\theta\| \leqslant \tau$, $\|\theta\| < \tau$, $\|\theta\| \geqslant \tau$ and $\|\theta\| > \tau$. We hereafter say that we test the norm of θ from above (resp. from below) τ when the tested hypothesis is either $\|\theta\| \leqslant \tau$ or $\|\theta\| < \tau$ (resp. $\|\theta\| \geqslant \tau$ or $\|\theta\| > \tau$). When there is no need to specify whether SNT is from above or from below, the interval involved in the hypothesis to test will be denoted by $\mathcal{I}_\tau$. Therefore, SNT of the deterministic signal θ, either from above or from below tolerance τ, is the testing of the composite hypothesis $\|\theta\| \in \mathcal{I}_\tau$ with $\theta \in \mathbb{R}^d$. Of course, for the problem to be meaningful, it is assumed that $\mathcal{I}_\tau$ and $\mathcal{I}_\tau^c$ are non-empty sets, where $\mathcal{I}_\tau^c$ henceforth denotes the complementary set $[0,\infty) \setminus \mathcal{I}_\tau$ of $\mathcal{I}_\tau$ in $[0,\infty)$. In

SNT from above (resp. from below), $\mathcal{I}_\tau$ can thus be any of the two intervals $[0,\tau]$ and $[0,\tau)$ (resp. any of the two intervals $[\tau,\infty)$ and (τ,∞)) when $\tau > 0$ and cannot be but $\{0\}$ (resp. $(0,\infty)$) when $\tau = 0$. For the sake of shortening the notation, we write $[0,\tau]\!\![$ (resp. $]\!\!]\tau,\infty)$) to designate any of the intervals $[0,\tau]$ and $[0,\tau)$ (resp. any of the two intervals $[\tau,\infty)$ and (τ,∞)), without specifying which of these two intervals is actually concerned and without recalling that this interval must be non-empty.

The results stated in this section rely on the following usual definitions [1]. Let $\mathcal{T}$ be some test. First, the *size of* $\mathcal{T}$ *for testing* $\|\theta\| \in \mathcal{I}_\tau$ *with* $\theta \in \mathbb{R}^d$ is defined by

$$\alpha(\mathcal{T}) = \sup_{\|\theta\| \in \mathcal{I}_\tau} \beta_\theta(\mathcal{T}). \tag{5}$$

Given $\gamma \in [0,1]$, $\mathcal{T}$ is said to have level (resp. size) γ for testing $\|\theta\| \in \mathcal{I}_\tau$ with $\theta \in \mathbb{R}^d$ if $\alpha(\mathcal{T}) \leqslant \gamma$ (resp. $\alpha(\mathcal{T}) = \gamma$). Hereafter, $\mathcal{K}_\gamma$ denotes the class of those tests $\mathcal{T}$ such that $\alpha(\mathcal{T}) \leqslant \gamma$. Second, the *power of* $\mathcal{T}$ *for testing the norm of* θ *with respect to* $\mathcal{I}_\tau$ is defined as the value of the power function $\beta_\theta(\mathcal{T})$ for θ such that $\|\theta\| \in \mathcal{I}_\tau^c$. The power of test $\mathcal{T}$ is thus the restriction of the power function of $\mathcal{T}$ to vectors θ with norms in $\mathcal{I}_\tau^c$. According to the standard definition of unbiased tests, an unbiased test for testing $\|\theta\| \in \mathcal{I}_\tau$ with $\theta \in \mathbb{R}^d$ is any test $\mathcal{T}$ such that $\beta_\theta(\mathcal{T}) \geqslant \alpha(\mathcal{T})$ for any $\theta \in \mathbb{R}^d$ such that $\|\theta\| \in \mathcal{I}_\tau^c$. Transposing standard terminology in statistical inference to SNT in the deterministic case, we put the following definition.

Definition 1 A test $\mathcal{T}$ is said to be *UMP with size* γ *within some class* $\mathcal{K}' \subset \mathcal{K}_\gamma$ of tests for testing $\|\theta\| \in \mathcal{I}_\tau$ with $\theta \in \mathbb{R}^d$ if **(i)** $\mathcal{T} \in \mathcal{K}'$, **(ii)** $\alpha(\mathcal{T}) = \gamma$ and **(iii)** $\beta_\theta(\mathcal{T}) \geqslant \beta_\theta(\mathcal{T}')$ for any $\mathcal{T}' \in \mathcal{K}'$ and any θ such that $\|\theta\| \in \mathcal{I}_\tau^c$. In particular, if there exists a UMP test with size γ within the class of all unbiased (resp. invariant) tests with level γ for testing $\|\theta\| \in \mathcal{I}_\tau$ with $\theta \in \mathbb{R}^d$, this test is said to be *UMP unbiased (UMPU)* (resp. *UMP invariant (UMPI)*) with size γ. If there exists a UMP test with size γ within the class of all possible tests with level γ for a given SNT problem, we simply say that this test is UMP with size γ.

On the basis of the previous definitions and material, three results are established below. They follow from standard ones in composite hypothesis testing, such as those given in [1]. More specifically, the first two concern the one-dimensional case only. In contrast, the third one — namely theorem 1 — establishes the existence, for any given dimension and any given level $\gamma \in (0,1)$, of thresholding tests that have size γ and that are uniformly most powerful among the tests with spherically invariant power function. These thresholding tests are said to be UMP-SIP with size γ. They are also UMPI since thresholding tests are basically invariant and invariant tests have spherically invariant power function.

Proposition 1 *Given some level* $\gamma \in (0,1)$ *and any* $\tau \in [0,\infty)$, *any thresholding test from below with threshold height* $\lambda_{1-\gamma}(\tau)$ *is UMP with size* γ *for testing* $\|\theta\| \in]\!\!]\tau,\infty)$ *with* $\theta \in \mathbb{R}^d$.

PROOF: The existence of a UMP test with size γ for testing the norm of θ from below tolerance τ simply follows from [1, Theorem 3.7.1] since the Gaussian distribution belongs to the one-parameter exponential family. It suffices to show that this test is actually the thresholding test from below with threshold height $\lambda_{1-\gamma}(\tau)$. According to [1, Theorem 3.7.1], the UMP test at hand is given by

$$\mathcal{T}(y) = \begin{cases} 0 & \text{if} \quad y < c_1 \text{ or } y > c_2 \\ 1 & \text{if} \quad c_1 < y < c_2 \end{cases}, \tag{6}$$

where $c_1 < c_2$ are determined so as to verify the equalities

$$\mathrm{P}\left[\, c_1 < -\tau + X < c_2 \,\right] = \mathrm{P}\left[\, c_1 < \tau + X < c_2 \,\right] = \gamma. \tag{7}$$

By setting $m = (c_1 + c_2)/2$ and $\Delta = (c_2 - c_1)/2$, it follows that $\left[\, c_1 < X - \tau < c_2 \,\right] = \left[\, |X - \tau - m| < \Delta \,\right]$ and that $\left[\, c_1 < X + \tau < c_2 \,\right] = \left[\, |X + \tau - m| < \Delta \,\right]$. Therefore, according to (1), the values c_1 and c_2 must satisfy $\mathcal{R}(|m + \tau|, \Delta) = \mathcal{R}(|\tau - m|, \Delta)$. The strict decreasingness of $\mathcal{R}(\cdot, \Delta)$ guaranteed by lemma 1 implies that $|m+\tau| = |\tau - m|$. Because $\tau > 0$, we must have $m = 0$ so that test $\mathcal{T}$ defined by (6) is necessarily the thresholding test from below with threshold $c_2 = -c_1$. It now follows from (1) and (7) that $\mathcal{R}(\tau, c_2) = \gamma$. Therefore, according to statement (i) of lemma 2, $c_2 = \lambda_{1-\gamma}(\tau)$ and the proof is complete. ■

For testing the norm of a deterministic signal from above, there is no UMP test. However, the following proposition states the existence of a uniformly most powerful unbiased (UMPU) test.

Proposition 2 *Given some level $\gamma \in (0, 1)$ and any $\tau \in [0, \infty)$, any thresholding test from above with threshold height $\lambda_\gamma(\tau)$ is UMPU with size γ for testing $\|\theta\| \in [0, \tau]\!]$ with $\theta \in \mathbb{R}^d$.*

PROOF: The existence of a UMPU test with size γ for testing the norm of $\theta \in \mathbb{R}$ from above tolerance τ follows from [1, Eqs. (4.2) & (4.3), section 4.2]. The proving that this test is actually the thresholding test from above with threshold height $\lambda_\gamma(\tau)$ mimics that of the preceeding proposition and is left to the reader. ■

The previous results are limited to the one-dimensional case and concern two differient criteria. It is desirable to obtain a result that holds for any dimension and optimizes a unique criterion. Basically, the problem is spherically invariant — or invariant under the action of the orthogonal group $\mathbf{O}_d$ in $\mathbb{R}^d$ — in the standard sense [1, Chapter 6, Section 6.1]. Indeed, for any given element g of $\mathbf{O}_d$, the noise probability distribution satisfies $\mathrm{P}\left[\, \|X\| \in B \,\right] = \mathrm{P}\left[\, \|gX\| \in B \,\right]$ for any Borel set of $\mathbb{R}^d$ and the hypothesis remains unchanged when the signal is $g\theta$ instead of θ. Therefore, it is natural to seek a UMPI test with level equal to some specified $\gamma \in (0, 1)$, that is, a UMP test within the class of those tests that are invariant under the group $\mathbf{O}_d$ and whose level is γ. If such a UMPI test exists, it follows from [1, Lemma 6.2.1] that this test must be a function of $\|\cdot\|$, which a maximal invariant of $\mathbf{O}_d$. It turns out that such a UMPI test actually exists and is a thresholding test for the following reasons. The group $\mathbf{O}_d$ leaves invariant the hypothesis $\|\theta\| \in \mathcal{I}_\tau$ to test. Since the norm $\|\cdot\|$ is a maximal invariant of $\mathbf{O}_d$, so is $\|\cdot\|^2$. Moreover, $\|Y\|^2$ is chi-2 distributed with d degrees of freedom and non-centrality parameter $\mu = \|\theta\|^2$. It then follows from [1, Theorem 6.2.1] — or [2, Theorem 1, Sec. 47, chapter III] — that the invariant tests for testing $\|\theta\| \in \mathcal{I}_\tau$ when we observe $Y \sim \mathcal{N}(\theta, \mathbf{I}_d)$ reduce to the tests for testing $\mu \in \{x^2 : x \in \mathcal{I}_\tau\}$ on the basis of $\|Y\|^2$. According to [5] or corollary 1 of appendix IV, the non-central chi-2 distribution has monotone likelihood ratio. The existence of a UMP test for testing $\mu \in \{x^2 : x \in \mathcal{I}_\tau\}$ is then a consequence of the Karlin-Rubin theorem [1, Theorem 3.4.1]. This UMP test is therefore UMPI for testing $\|\theta\| \in \mathcal{I}_\tau$ on the basis of the initial observation Y. It remains to prove that this UMPI test is actually the threshold test from above (resp. from below) with threshold height $\lambda_\gamma(\tau)$ (resp. $\lambda_{1-\gamma}(\tau)$) when SNT is from above (resp. from below) tolerance τ. This is achieved by proceeding as in the proof of proposition 1.

In fact, more can be said about thresholding tests in SNT of a deterministic signal. To this end, we consider the tests with spherically invariant power function. Although the following definition for such tests is straightforward and could be omitted, we however prefer making it, so as to introduce the terminology chosen throughout to designate such tests, especially with regard to the contents of section 5.

Definition 2 A test $\mathcal{T}$ is said to have *spherically invariant power function* (SIP*fun*) if $\beta_\theta(\mathcal{T}(g)) = \beta_\theta(\mathcal{T})$ for any element g of $\mathbf{O}_d$ and any $\theta \in \mathbb{R}^d$, where $\mathcal{T}(g)$ is the composite map $\mathcal{T} \circ g$. The class of the tests with SIP*fun* is hereafter denoted by $\mathcal{K}_{\mathrm{SIP}fun}$.

Note that the tests with SIP*fun* are also the tests $\mathcal{T}$ whose power function $\beta_\theta(\mathcal{T})$ is a function of $\|\theta\|$ and, thus, constant on every sphere with radius $\rho \in (0, \infty)$, which are the orbits of the orthogonal group in $\mathbb{R}^d$. Recall that if invariant tests have necessarily SIP*fun*, the converse is however not true [1, Chapter 6, pp. 227 – 228]. We can now state the main result of this section.

Theorem 1 *Given some level $\gamma \in (0, 1)$, any thresholding test from above (resp. from below) with threshold height $\lambda_\gamma(\tau)$ (resp. $\lambda_{1-\gamma}(\tau)$) is unbiased and UMP with size γ within $\mathcal{K}_{SIP\mathrm{fun}} \cap \mathcal{K}_\gamma$ — we say that this test is UMP-SIP with size γ — for testing $\left[\, \|\theta\| \in \mathcal{I}_\tau \,\right]$ with $\theta \in \mathbb{R}^d$ and $\mathcal{I}_\tau = [0, \tau[\![$ (resp. $\mathcal{I}_\tau =]\!]\tau, \infty)$).*

PROOF: The fact that the thresholding tests specified in the statement are UMP-SIP with size γ for testing $\left[\, \|\theta\| \in \mathcal{I}_\tau \,\right]$ with $\theta \in \mathbb{R}^d$ straighforwardly follows from theorems 2 and 3 established below. The only thing we prove here is the unbiasedness of these tests. In fact, we prove this unbiasedness in SNT from above τ only, for the proving in SNT from below τ is similar and can be left to the reader. We thus consider the problem of testing $\left[\, \|\theta\| \in \mathcal{I}_\tau \,\right]$ with $\theta \in \mathbb{R}^d$ and $\mathcal{I}_\tau = [0, \tau[\![$. Let $\mathcal{T}_{\lambda^*}$ be any thresholding test from above with threshold height $\lambda^* = \lambda_\gamma(\tau)$ so that $1 - \mathcal{R}(\tau, \lambda^*) = \gamma$. According to (1), $\beta_\theta(\mathcal{T}_{\lambda^*}) = 1 - \mathcal{R}(\|\theta\|, \lambda^*)$ for any $\theta \in \mathbb{R}^d$. On the other hand, it follows from lemma 1 that $1 - \mathcal{R}(\cdot, \lambda^*)$ increases strictly. Therefore, we derive from the foregoing that $\beta_\theta(\mathcal{T}_{\lambda^*}) \geqslant \gamma$ for any θ such that $\|\theta\| \in \mathcal{I}_\tau^c$. Thence, the unbiasedness of $\mathcal{T}_{\lambda^*}$. ■

4.2 Connection to Wald's theory of tests with uniformly best constant power

We now show that theorem 1 embraces Wald's proposition [26, Section 6, Proposition III, p. 450] about tests with uniformly best constant power (UBCP) for testing the mean of a Gaussian distributed random vector. To begin with, we briefly recall Wald's definition [26, Definition III, Section 6, p. 450]. Then, we present the hypothesis testing problem addressed by Wald's proposition before stating and proving [26, Section 6, Proposition III, p. 450], as a consequence of theorem 1. Wald's definition or criterion is the following one.

Definition 3 [Wald's UBCP tests] Let Y be a d-dimensional real random vector whose distribution belongs to a given class $\{\mathrm{P}_\theta : \theta \in \varpi\}$, where ϖ is some parameter space. For any given $\theta \in \varpi$, let $\mathrm{P}_\theta\left[\mathcal{T}(Y) = 1\right]$ stand for the probability value $\mathrm{P}\left[\mathcal{T}(Y) = 1\right]$ when the distribution of Y is P_θ. Let $\left\{\Upsilon_\rho : \rho \in \mathfrak{I}\right\}$ be a family of surfaces in ϖ where $\mathfrak{I}$ is some index set. For testing $\mathsf{H}_0 : \theta = \theta_0$ against $\mathsf{H}_1 : \theta \neq \theta_0$ where $\theta_0 \in \varpi$, a test $\mathcal{T}$ is said to be UBCP on $\left\{\Upsilon_\rho : \rho \in \mathfrak{I}\right\}$ if it satisfies the following two conditions:

(a) Test $\mathcal{T}$ has constant power function on every Υ_ρ, $\rho \in \mathfrak{I}$, in that, given any $\rho \in \mathfrak{I}$, $\mathrm{P}_\theta\left[\mathcal{T}(Y)=1\right] = \mathrm{P}_{\theta'}\left[\mathcal{T}(Y)=1\right]$ for any $\theta, \theta' \in \Upsilon_\rho \subset \varpi$.

(b) For any $\theta \in \varpi$, $\mathrm{P}_\theta\left[\mathcal{T}(Y)=1\right] \geqslant \mathrm{P}_\theta\left[\mathcal{T}'(Y)=1\right]$ for any test $\mathcal{T}'$ whose power is constant on every given Υ_ρ with $\rho \in \mathfrak{I}$ and such that $\mathrm{P}_{\theta_0}\left[\mathcal{T}(Y)=1\right] = \mathrm{P}_{\theta_0}\left[\mathcal{T}'(Y)=1\right]$.

Wald's result [26, Section 6, Proposition III, p. 450] can then be rewritten in the following form, with no loss of generality. We prove it as a consequence of theorem 1.

Proposition 3 [26, Section 6, Proposition III] Let Y be some random d-dimensional random vector whose distribution belongs to the family $\{\mathcal{N}(\theta, \mathbf{I}_d) : \theta \in \mathbb{R}^d\}$. For testing $\mathsf{H}_0 : \mathbb{E}Y = 0$ against $\mathsf{H}_1 : \mathbb{E}Y \neq 0$, any thresholding test from above whose threshold is positive is UBCP on the family of spheres $\rho\mathrm{S}^{d-1}$ with $\rho \in [0,\infty)$.

PROOF: Let $\mathcal{T}_\eta$ be some thresholding test from above with threshold height $\eta > 0$. We must prove that **(1)** $\mathcal{T}_\eta$ has constant power on every sphere $\rho\mathrm{S}^{d-1}$ with $\rho \in [0,\infty)$ and **(2)** $\beta_\theta(\mathcal{T}_\eta) \geqslant \beta_\theta(\mathcal{T}')$ for any $\theta \in \mathbb{R}^d$ and any test $\mathcal{T}'$ with constant power on every sphere $\rho\mathrm{S}^{d-1}$ with $\rho \in [0,\infty)$ and such that $\beta_0(\mathcal{T}_\eta) = \beta_0(\mathcal{T}')$.

First, the tests whose power is constant on every given sphere $\rho\mathrm{S}^{d-1}$, $\rho \geqslant 0$, are exactly the tests with SIP*fun*. Therefore, as any thresholding test, $\mathcal{T}_\eta$ has SIP*fun* and, thus, constant power on every sphere $\rho\mathrm{S}^{d-1}$ with $\rho \geqslant 0$, which proves **(1)**.

Second, the problem of testing H_0 against H_1 is equivalent to the SNT problem of testing $\|\mathbb{E}Y\|$ from above tolerance $\tau = 0$. This and (5) imply that

$$\beta_0(\mathcal{T}) = \alpha(\mathcal{T}), \tag{8}$$

for any test $\mathcal{T}$. Since lemma 2 guarantees the existence of a unique $\gamma \in (0,1)$ such that $\eta = \lambda_\gamma(0)$, it follows from theorem 1 that $\mathcal{T}_\eta$ is UMP-SIP for testing $\|\theta\| = 0$ and (8) implies that

$$\beta_0(\mathcal{T}_\eta) = \alpha(\mathcal{T}_\eta) = \gamma. \tag{9}$$

Let $\mathcal{T}'$ be any other test with constant power on every sphere $\rho\mathrm{S}^{d-1}$ with $\rho \geqslant 0$ and such that $\beta_0(\mathcal{T}_\eta) = \beta_0(\mathcal{T}')$. This test $\mathcal{T}'$ has thus SIP*fun* and, according to (8) and (9), is such that $\alpha(\mathcal{T}') = \alpha(\mathcal{T}_\eta) = \gamma$. Thereby, $\mathcal{T}_\eta$ and $\mathcal{T}'$ are both elements of $\mathcal{K}_{\mathrm{SIP}fun} \cap \mathcal{K}_\gamma$. Since $\mathcal{T}_\eta$ is UMP-SIP with size γ for testing $\|\theta\| = 0$, $\beta_\theta(\mathcal{T}_\eta) \geqslant \beta_\theta(\mathcal{T}')$ and **(2)** holds true. ■

According to theorem 1 and proposition 3, given any $\gamma \in (0,1)$, any thresholding test from above with threshold height equal to $\lambda_\gamma(0)$ is UBCP on the family of spheres $\rho\mathrm{S}^{d-1}$ with $\rho \in [0,\infty)$ and UMP-SIP with size γ for testing $\theta = 0$ with $\theta \in \mathbb{R}^d$. Throughout the rest of the paper, any thresholding test with threshold height $\lambda_\gamma(0)$ will be called *Wald's test with size* γ.

5 Testing the norm of a random signal

5.1 Mathematical statement

All the random vectors and variables are assumed to be defined on the same probability space $(\Omega, \mathcal{B}, \mathrm{P})$. As usual, we write (a-s) for almost surely. The set of all d-dimensional real random vectors defined on $(\Omega, \mathcal{B})$ and valued, thus, in $\mathbb{R}^d$, is hereafter denoted by $\mathcal{M}(\Omega, \mathbb{R}^d)$. Throughout this section, the observation is $Y = \Theta + X$

where Θ is an unknown element of $\mathcal{M}(\Omega,\mathbb{R}^d)$ and $X \sim \mathcal{N}(0,\mathbf{I}_d)$ is standard Gaussian noise. Given any $Z \in \mathcal{M}(\Omega,\mathbb{R}^d)$ (resp. any random variable), $\mathrm{P}Z^{-1}$ stands for the probability distribution of Z, that is the probability measure defined for any Borel subset B of $\mathbb{R}^d$ (resp. $\mathbb{R}$) by $\mathrm{P}Z^{-1}(B) = \mathrm{P}\left[\, Z \in B \,\right]$.

The SNT problem in the random case can be posed as follows. Given some non-negative real number τ and some elementary event $\omega \in \Omega$, we want to know whether $\|\Theta(\omega)\| \leqslant \tau$ or not, when we are given $Y(\omega)$ and the probability distribution of Θ is unknown. By analogy with standard terminology in statistical inference, we say that we test the event $\left[\, \|\Theta\| \leqslant \tau \,\right]$. In fact, 3 other events can actually be tested in SNT of a random signal. These possible events are $\left[\, \|\Theta\| < \tau \,\right]$, $\left[\, \|\Theta\| > \tau \,\right]$ and $\left[\, \|\Theta\| \geqslant \tau \,\right]$. The results established below do not depend on whether the inequality is strict or not in the event to test. This follows again from the absolute continuity of the probability distribution of $\Theta + X$ with respect to Lebesgue's measure in $\mathbb{R}^d$. All the possible events that can be tested in SNT in the random case can be written in the form $\left[\, \|\Theta\| \in \mathcal{I}_\tau \,\right]$, where $\mathcal{I}_\tau$ is any of the four intervals $[0,\tau]$, $[0,\tau)$, $[\tau,\infty)$ and (τ,∞). Of course, SNT in the random case is of actual interest when the signal is assumed to be an element of the set ϑ_τ of those d-dimensional real random vectors Θ such that $\mathrm{P}\left[\, \|\Theta\| \in \mathcal{I}_\tau \,\right] \in (0,1)$, which implies that $\mathcal{I}_\tau$ and $\mathcal{I}_\tau^c = [0,\infty) \setminus \mathcal{I}_\tau$ are non-empty sets, as in section 4. Thereby, our focus will hereafter be the problem of testing the event $[\, \|\Theta\| \in \mathcal{I}_\tau \,]$ with $\Theta \in \vartheta_\tau$, under the necessary assumption that neither $\mathcal{I}_\tau$ nor $\mathcal{I}_\tau^c$ is empty. This necessary assumption is implicit throughout. As in the deterministic case, SNT in the random case is said to be from above (resp. from below) tolerance τ when $\mathcal{I}_\tau$ is any of the two intervals $[0,\tau]$ and $[0,\tau)$ (resp. any of the two intervals $[\tau,\infty)$ and (τ,∞)). In the sequel, we keep on using the notation $[0,\tau]\!\![$ (resp. $]\!\!]\tau,\infty)$) to designate any of the intervals $[0,\tau]$ and $[0,\tau)$ (resp. any of the two intervals $[\tau,\infty)$ and (τ,∞)).

Basically, testing $\left[\, \|\Theta\| \in \mathcal{I}_\tau \,\right]$ with $\Theta \in \vartheta_\tau$ amounts to choosing some map of Ω into $\{0,1\}$ so that, for every $\omega \in \Omega$, the value returned by this map is the decision on whether $\|\Theta(\omega)\|$ is an element of $\mathcal{I}_\tau$ or not. Similarly to standard terminology in statistical inference, if this decision assigned to a given $\omega \in \Omega$ is 0 (resp. 1), the event $\left[\, \|\Theta\| \in \mathcal{I}_\tau \,\right]$ is said to be accepted (resp. rejected). Of course, there are infinitely many possible choices for maps of Ω into $\{0,1\}$. In the sequel, we restrict our attention to the rather natural class of the composite maps $\mathcal{T} \circ Y = \mathcal{T}(Y)$ where $\mathcal{T}$ is any test, that is, any measurable map of $\mathbb{R}^d$ into $\{0,1\}$. We then assess the performance of a given test $\mathcal{T}$ for testing $\left[\, \|\Theta\| \in \mathcal{I}_\tau \,\right]$ with $\Theta \in \vartheta_\tau$ via the following two quantities, whose definitions extend the standard notions [1] of size and power. To begin with, the *size of* $\mathcal{T}$ *for testing the norm of a given* $\Theta \in \vartheta_\tau$ *with respect to* $\mathcal{I}_\tau$ is defined as the conditional

$$\alpha_\Theta^{\vartheta_\tau}(\mathcal{T}) = \mathrm{P}\left[\, \mathcal{T}(\Theta + X) = 1 \,\middle|\, \|\Theta\| \in \mathcal{I}_\tau \,\right]. \tag{10}$$

and the *size of* $\mathcal{T}$ *for testing* $\left[\, \|\Theta\| \in \mathcal{I}_\tau \,\right]$ *with* $\Theta \in \vartheta_\tau$ is defined by

$$\alpha^{\vartheta_\tau}(\mathcal{T}) = \sup_{\Theta \in \vartheta_\tau} \alpha_\Theta^{\vartheta_\tau}(\mathcal{T}). \tag{11}$$

Test $\mathcal{T}$ is then said to have level (resp. size) $\gamma \in [0,1]$ for testing $\left[\, \|\Theta\| \in \mathcal{I}_\tau \,\right]$ with $\Theta \in \vartheta_\tau$ if $\alpha^{\vartheta_\tau}(\mathcal{T}) \leqslant \gamma$ (resp. $\alpha^{\vartheta_\tau}(\mathcal{T}) = \gamma$). Given $\gamma \in [0,1]$, $\mathcal{K}_\gamma^{\vartheta_\tau}$ will henceforth denote the class of those tests $\mathcal{T}$ such that $\alpha^{\vartheta_\tau}(\mathcal{T}) \leqslant \gamma$. Second, the *power of* $\mathcal{T}$ *for testing the norm of a given* $\Theta \in \vartheta_\tau$ *with respect to* $\mathcal{I}_\tau$ is defined as the conditional

$$\beta_\Theta^{\vartheta_\tau}(\mathcal{T}) = \mathrm{P}\left[\, \mathcal{T}(\Theta + X) = 1 \,\middle|\, \|\Theta\| \in \mathcal{I}_\tau^c \,\right]. \tag{12}$$

A test $\mathcal{T}$ with level γ for testing $\left[\,\|\Theta\| \in \mathcal{I}_\tau\,\right]$ with $\Theta \in \vartheta_\tau$ is said to be unbiased if, for any given $\Theta \in \vartheta_\tau$, $\beta_\Theta^{\vartheta_\tau}(\mathcal{T}) \geqslant \gamma$. Given two tests $\mathcal{T}$ and $\mathcal{T}'$ with same level for testing $\left[\,\|\Theta\| \in \mathcal{I}_\tau\,\right]$ with $\Theta \in \vartheta_\tau$, $\mathcal{T}$ is said to be more powerful that $\mathcal{T}'$ for testing the norm of a given $\Theta \in \vartheta_\tau$ with respect to $\mathcal{I}_\tau$ if $\beta_\Theta^{\vartheta_\tau}(\mathcal{T}) \geqslant \beta_\Theta^{\vartheta_\tau}(\mathcal{T}')$. The following lemma emphasizes that the above notions of size and power in the random case relate to those of section 4 dedicated to the deterministic case.

Lemma 3 Let $\mathcal{T}$ be some test. We have:

(i) given any $\theta \in \mathbb{R}^d$ with norm in $\mathcal{I}_\tau$, $\beta_\theta(\mathcal{T}) \leqslant \alpha(\mathcal{T}) \leqslant \alpha^{\vartheta_\tau}(\mathcal{T})$;

(ii) given any $\theta' \in \mathbb{R}^d$ with norm in $\mathcal{I}_\tau^c$, $\beta_{\theta'}(\mathcal{T}) = \beta_\Theta^{\vartheta_\tau}(\mathcal{T})$ for any $\Theta \in \vartheta_\tau$ such that $\Theta = \varepsilon\theta' + (1-\varepsilon)\theta$ where $\theta \in \mathbb{R}^d$ has norm in $\mathcal{I}_\tau$ and ε is a Bernouilli distributed random variable valued in $\{0,1\}$ with $\mathrm{P}\left[\,\varepsilon = 1\,\right] \in (0,1)$.

PROOF: Let θ and θ' be any two elements of $\mathbb{R}^d$ such that $\|\theta\| \in \mathcal{I}_\tau$ and $\|\theta'\| \in \mathcal{I}_\tau^c$. Let ε stand for some Bernouilli distributed random variable valued in $\{0,1\}$ with $\mathrm{P}\left[\,\varepsilon = 1\,\right] \in (0,1)$. The random vector $\Theta = (1-\varepsilon)\theta + \varepsilon\theta'$ is an element of ϑ_τ. Let $\mathcal{T}$ be some test. From (4) and (11), we derive that $\beta_\theta(\mathcal{T}) \leqslant \alpha^{\vartheta_\tau}(\mathcal{T})$. Since θ is arbitrarily chosen so that $\|\theta\| \in \mathcal{I}_\tau$, the second inequality in statement **(i)** follows from (5). Statement **(ii)** is a direct consequence of (12). ∎

Similarly to the deterministic case, our purpose is to pinpoint tests in $\mathcal{K}_\gamma^{\vartheta_\tau}$ whose power is optimal, with respect to a certain criterion, for testing the norms of the elements of ϑ_τ. Because of the next remark and comments, the criterion must necessarily concern a restricted family of tests.

Remark 1 The same type of reasoning as above makes it possible to derive that there is no UMP test with level $\gamma \in (0,1)$ for testing $\left[\,\|\Theta\| \in \mathcal{I}_\tau\,\right]$ with $\Theta \in \vartheta_\tau$. By UMP test with level γ for testing $\left[\,\|\Theta\| \in \mathcal{I}_\tau\,\right]$ with $\Theta \in \vartheta_\tau$, we mean some $\mathcal{T} \in \mathcal{K}_\gamma^{\vartheta_\tau}$ such that $\beta_\Theta^{\vartheta_\tau}(\mathcal{T}) \geqslant \beta_\Theta^{\vartheta_\tau}(\mathcal{T}')$ for any $\Theta \in \vartheta_\tau$ and any $\mathcal{T}' \in \mathcal{K}_\gamma^{\vartheta_\tau}$. Such a UMP test does not exist for the following reason. Let θ and θ' be any two elements of $\mathbb{R}^d$ such that $\|\theta\| \in \mathcal{I}_\tau$ and $\|\theta'\| \in \mathcal{I}_\tau^c$. Consider again any Bernouilli distributed random variable ε valued in $\{0,1\}$ such that $\mathrm{P}\left[\,\varepsilon = 1\,\right] \in (0,1)$ and construct the random vector $\Theta = (1-\varepsilon)\theta + \varepsilon\theta' \in \vartheta_\tau$. If a UMP test $\mathcal{T}$ existed within $\mathcal{K}_\gamma^{\vartheta_\tau}$, it follows from lemma 3 that this test would be most powerful with level γ to test $\mu = \theta$ against $\mu = \theta'$, when the observation is Gaussian distributed with mean μ and covariance matrix $\mathbf{I}_d$. The existence of such a most powerful test, independent of the arbitrarily chosen θ and θ' such that $\|\theta\| \in \mathcal{I}_\tau$ and $\|\theta'\| \in \mathcal{I}_\tau^c$, would then contradict the Neyman-Pearson lemma [1, Theorem 3.2.1, Sec. 3.2, p. 60].

In addition to the non-existence of UMP tests for SNT in the random case, neither standard general results — such as Karlin-Rubin's theorem [1, Theorem 3.4.1, p. 65, corollary 3.4.1, p. 67] or [1, Theorem 3.7.1, p. 81] — nor the results of section 4 apply to the SNT problem addressed in this section. The main reason is that the signal probability distribution is unknown. Thereby, standard arguments and results based on invariance cannot be used directly to reduce the problem. The invariance of both the problem and the noise distribution can, however, still be used through another criterion for optimality. This alternative criterion is that of definition 6 below and relies on the following conditional notion of power function.

5.2 Spherically-conditioned power function

Signal norm testing in the random case is an invariant problem in that, given any element g of the orthogonal group $\mathbf{O}_d$ in $\mathbb{R}^d$, $g\Theta + gX$ is also an element of ϑ_τ and $\left[\,\|\Theta\| \in \mathcal{I}_\tau\,\right] = \left[\,\|g\Theta\| \in \mathcal{I}_\tau\,\right]$. On the other hand, theorem 1 above has exhibited UMP-SIP tests, with level $\gamma \in (0,1)$, for testing $\left[\,\|\theta\| \in \mathcal{I}_\tau\,\right]$ with $\theta \in \mathbb{R}^d$, when the observation is $Y = \theta + X$. The natural question that arises at this stage is whether these UMP-SIP tests would not actually satisfy some additional invariance-based optimality, for testing $\left[\,\|\Theta\| \in \mathcal{I}_\tau\,\right]$ with $\Theta \in \vartheta_\tau$. Theorem 3 provides an affirmative answer to this question on the basis of the following definition, which extends the notion of test with SIP*fun*, so as to embrace the case of a random signal.

Definition 4 The *spherically-conditioned power function* (SCP*fun*) of a given test $\mathcal{T}$ is the map that assigns to each $\Theta \in \mathcal{M}(\Omega, \mathbb{R}^d)$ the unique element $\beta_\Theta\left(\mathcal{T} \,\middle|\, \|\Theta\| = \cdot\right) \in \mathrm{L}^1(\mathrm{P}\|\Theta\|^{-1})$ defined for every $\rho \in [0,\infty)$ by

$$\beta_\Theta\left(\mathcal{T} \,\middle|\, \|\Theta\| = \rho\right) = \mathrm{P}\left[\mathcal{T}(\Theta + X) = 1 \,\middle|\, \|\Theta\| = \rho\right].$$

With the notation of the previous definition, the basic property of $\beta_\Theta\left(\mathcal{T} \,\middle|\, \|\Theta\| = \cdot\right)$ is that

$$\mathrm{P}\left(\left[\mathcal{T}(\Theta + X) = 1\right] \cap B\right) = \int_B \beta_\Theta\left(\mathcal{T} \,\middle|\, \|\Theta\| = \rho\right) \mathrm{P}\|\Theta\|^{-1}(\mathrm{d}\rho) \tag{13}$$

for any Borel set B of $\mathbb{R}$. The SCP*fun* is analogous to the standard power function and, in fact, relates to it as follows. Let $\Theta = \varepsilon\theta + (1-\varepsilon)\theta'$ where θ and θ' are two elements of $\mathbb{R}^d$ such that $\|\theta'\| \neq \|\theta\|$ and ε stands for some Bernouilli distributed random variable valued in $\{0,1\}$, with $\mathrm{P}\left[\varepsilon = 1\right] \in (0,1)$. We then have $\beta_\theta(\mathcal{T}) = \beta_\Theta\left(\mathcal{T} \,\middle|\, \|\Theta\| = \|\theta\|\right)$.

As shown by the next equalities, the SCP*fun* of a test $\mathcal{T}$ also relates to the size and power of $\mathcal{T}$ for testing the norm of a given $\Theta \in \mathcal{I}_\tau$. First, for any $\Theta \in \vartheta_\tau$ and any Borel set B such that $\mathrm{P}\left[\,\|\Theta\| \in B\,\right] \neq 0$, Bayes's rule and (13) induce that

$$\mathrm{P}\left[\mathcal{T}(\Theta + X) = 1 \,\middle|\, \|\Theta\| \in B\right] = \frac{1}{\mathrm{P}\left[\|\Theta\| \in B\right]} \int_B \beta_\Theta\left(\mathcal{T} \,\middle|\, \|\Theta\| = \rho\right) \mathrm{P}\|\Theta\|^{-1}(\mathrm{d}\rho). \tag{14}$$

It then suffices to apply (14) to $\mathcal{I}_\tau$ and $\mathcal{I}_\tau^c$ to obtain that:

$$\alpha_\Theta^{\vartheta_\tau}(\mathcal{T}) = \frac{1}{\mathrm{P}\left[\|\Theta\| \in \mathcal{I}_\tau\right]} \int_{\mathcal{I}_\tau} \beta_\Theta\left(\mathcal{T} \,\middle|\, \|\Theta\| = \rho\right) \mathrm{P}\|\Theta\|^{-1}(\mathrm{d}\rho) \tag{15}$$

and

$$\beta_\Theta^{\vartheta_\tau}(\mathcal{T}) = \frac{1}{\mathrm{P}\left[\|\Theta\| \in \mathcal{I}_\tau^c\right]} \int_{\mathcal{I}_\tau^c} \beta_\Theta\left(\mathcal{T} \,\middle|\, \|\Theta\| = \rho\right) \mathrm{P}\|\Theta\|^{-1}(\mathrm{d}\rho). \tag{16}$$

Let $\mathcal{T}$ and $\mathcal{T}'$ be two elements of $\mathcal{K}_\gamma^{\vartheta_\tau}$, that is, two tests with same level γ for testing $\left[\,\|\Theta\| \in \mathcal{I}_\tau\,\right]$ with $\Theta \in \vartheta_\tau$. If $\beta_\Theta\left(\mathcal{T} \,\middle|\, \|\Theta\| = \rho\right) \geqslant \beta_\Theta\left(\mathcal{T}' \,\middle|\, \|\Theta\| = \rho\right)$ for a given $\Theta \in \vartheta_\tau$ and $\mathrm{P}\|\Theta\|^{-1}$ – almost every $\rho \in \mathcal{I}_\tau^c$, $\mathcal{T}$ is more powerful than $\mathcal{T}'$ for testing the norm of $\Theta \in \vartheta_\tau$ with respect to $\mathcal{I}_\tau$. Therefore, there is no test $\mathcal{T} \in \mathcal{K}_\gamma^{\vartheta_\tau}$ such that, for all $\mathcal{T}' \in \mathcal{K}_\gamma^{\vartheta_\tau}$ and all $\Theta \in \vartheta_\tau$, $\beta_\Theta\left(\mathcal{T} \,\middle|\, \|\Theta\| = \rho\right) \geqslant \beta_\Theta\left(\mathcal{T}' \,\middle|\, \|\Theta\| = \rho\right)$ for $\mathrm{P}\|\Theta\|^{-1}$ – almost every $\rho \in \mathcal{I}_\tau^c$. Indeed, if such a test $\mathcal{T}$ existed, it follows from (16) that this test would be, in fact, UMP with level γ for testing $\left[\|\Theta\| \in \mathcal{I}_\tau\right]$ with $\Theta \in \vartheta_\tau$, a contradiction with remark 1. Hence, we extend the notions of tests with SIP*fun* to come up with a suitable criterion that can be optimized to test $\left[\,\|\Theta\| \in \mathcal{I}_\tau\,\right]$ with $\Theta \in \vartheta_\tau$.

5.3 Tests with uniformly best invariant spherically-conditioned power

On the one hand, the notion of tests with SIP*fun* extend that of invariant tests. On the other hand, the notion of SCP*fun* extends that of power function. An SCP*fun*-based definition of invariant tests is then introduced. This definition extends that of tests with SIP*fun* and the properties of these ones can then be extended to SNT in the random case.

Definition 5 Let $\mathcal{T}$ be some test. Given $\Theta \in \mathcal{M}(\Omega, \mathbb{R}^d)$, $\mathcal{T}$ is said to have Θ-*invariant* SCP*fun* — we say that $\mathcal{T}$ has Θ-ISCP*fun* — over $\rho\mathrm{S}^{d-1}$ with $\rho \in [0, \infty)$ if

$$\beta_\Theta\left(\mathcal{T} \,\middle|\, \|\Theta\| = \rho\right) = \beta_\theta(\mathcal{T})$$

for any $\theta \in \rho\mathrm{S}^{d-1}$.

An immediate consequence of the preceeding definition is that the power function of any test $\mathcal{T}$ with Θ-ISCP*fun* over a given sphere $\rho\mathrm{S}^{d-1}$ is constant on this same sphere, so that $\beta_\theta(\mathcal{T}) = \beta_{\theta'}(\mathcal{T})$ for any $\theta, \theta' \in \rho\mathrm{S}^{d-1}$. The following proposition could have been our definition for tests with Θ-ISCP*fun*.

Proposition 4 Given some test $\mathcal{T}$ and some $\Theta \in \mathcal{M}(\Omega, \mathbb{R}^d)$, $\mathcal{T}$ has Θ-ISCP*fun* over $\rho\mathrm{S}^{d-1}$ with $\rho \in [0, \infty)$ if and only if $\beta_\Theta\left(\mathcal{T} \,\middle|\, \|\Theta\| = \rho\right) = \mathrm{P}\left[\mathcal{T}(\Xi + X) = 1\right]$ for any $\Xi \in \mathcal{M}(\Omega, \mathbb{R}^d)$ independent of X and such that $\|\Xi\| = \rho$ (a-s).

PROOF: Given some $\Theta \in \mathcal{M}(\Omega, \mathbb{R}^d)$, suppose that $\mathcal{T}$ is some test with Θ-ISCP*fun* over $\rho\mathrm{S}^{d-1}$ with $\rho \in [0, \infty)$. Let Ξ be some element of $\mathcal{M}(\Omega, \mathbb{R}^d)$, independent of X and such that $\|\Xi\| = \rho$ (a-s). We therefore have

$$\mathrm{P}\left[\mathcal{T}(\Xi + X) = 1\right] = \int_{\rho\mathrm{S}^{d-1}} \mathrm{P}\left[\mathcal{T}(\Xi + X) = 1 \,\middle|\, \Xi = \xi\right] \left(\mathrm{P}\Xi^{-1}\right)(\mathrm{d}\xi).$$

From the independence of Ξ and X, we obtain that

$$\mathrm{P}\left[\mathcal{T}(\Xi + X) = 1\right] = \int_{\rho\mathrm{S}^{d-1}} \mathrm{P}\left[\mathcal{T}(\xi + X) = 1\right] \left(\mathrm{P}\Xi^{-1}\right)(\mathrm{d}\xi).$$

Since $\mathcal{T}$ has Θ-ISCP*fun* over $\rho\mathrm{S}^{d-1}$, the integrand of the right hand side (rhs) in the equality above is constant and equal to $\beta_\Theta\left(\mathcal{T} \,\middle|\, \|\Theta\| = \rho\right)$. The direct implication stated by the proposition follows.

Conversely, suppose that $\beta_\Theta\left(\mathcal{T} \,\middle|\, \|\Theta\| = \rho\right) = \mathrm{P}\left[\mathcal{T}(\Xi + X) = 1\right]$ for any $\Xi \in \mathcal{M}(\Omega, \mathbb{R}^d)$, independent of X and such that $\|\Xi\| = \rho$ (a-s). It then suffices to choose $\Xi = \theta$ (a-s) where θ is any element of $\rho\mathrm{S}^{d-1}$ to obtain that $\beta_\Theta\left(\mathcal{T} \,\middle|\, \|\Theta\| = \rho\right) = \beta_\theta(\mathcal{T})$, which concludes the proof. ■

The following proposition relates the notion of test with Θ-ISCP*fun* to the notion of test with SIP*fun*. The criterion that will be considered in definition 6 for testing the norm of a random signal basically relies on this result, which indicates how to extend the notion of UMP-SIP test to the random case. For readiness sake, we recall that, given $\Theta \in \mathcal{M}(\Omega, \mathbb{R}^d)$, a support of $\mathrm{P}\|\Theta\|^{-1}$ is any measurable subset $\mathfrak{D}$ of $[0, \infty)$ such that $\mathrm{P}\|\Theta\|^{-1}(\mathfrak{D}) = \mathrm{P}\left[\|\Theta\| \in \mathfrak{D}\right] = 1$. Note that $[0, \infty)$ is a support of $\mathrm{P}\|\Theta\|^{-1}$ for any $\Theta \in \mathcal{M}(\Omega, \mathbb{R}^d)$. On the other hand, for any $\rho \in [0, \infty)$, $\mathfrak{D} = \{\rho\}$ is a support of $\mathrm{P}\|\Theta\|^{-1}$ for any Θ distributed on $\rho\mathrm{S}^{d-1}$.

Proposition 5 Let $\mathcal{T}$ be some test. We have:
(i) $\mathcal{T}$ has SIP*fun* if and only if, for any $\Theta \in \mathcal{M}(\Omega, \mathbb{R}^d)$, there exists a support $\mathfrak{D}$ of $\mathrm{P}\|\Theta\|^{-1}$ such that $\mathcal{T}$ has Θ-ISCP*fun* over any sphere with radius in $\mathfrak{D}$.
(ii) if $\mathcal{T}$ has SIP*fun* and level $\gamma \in [0,1]$ for testing $\left[\, \|\theta\| \in \mathcal{I}_\tau \,\right]$ with $\theta \in \mathbb{R}^d$, then $\mathcal{T}$ has level γ for testing $\left[\, \|\Theta\| \in \mathcal{I}_\tau \,\right]$ with $\Theta \in \vartheta_\tau$.

PROOF: See appendix III. ∎

The previous result directly induce the following properties of thresholding tests.

Lemma 4 Let η be some non-negative real number and $\mathcal{T}_\eta$ be some thresholding test with threshold height η.

(i) For any $\Theta \in \vartheta_\tau$, there exists a support $\mathfrak{D}$ of $\mathrm{P}\|\Theta\|^{-1}$ such that, for every $\rho \in \mathfrak{D}$, $\mathcal{T}_\eta$ has Θ-ISCP*fun* over $\rho \mathrm{S}^{d-1}$ so that

$$\beta_\Theta\left(\mathcal{T}_\eta \,\middle|\, \|\Theta\| = \rho\right) = 1 - \mathcal{R}(\rho, \eta) \tag{17}$$

if the thresholding is from above and

$$\beta_\Theta\left(\mathcal{T}_\eta \,\middle|\, \|\Theta\| = \rho\right) = \mathcal{R}(\rho, \eta) \tag{18}$$

if the thresholding is from below. These equalities are equivalent to

$$\mathrm{P}\left[\, \|\Theta + X\| \in [0, \eta]\!] \,\middle|\, \|\Theta\| = \rho \,\right] = \mathcal{R}(\rho, \eta). \tag{19}$$

(ii) The size of $\mathcal{T}_\eta$ for testing the norm of a given $\Theta \in \vartheta_\tau$ with respect to $\mathcal{I}_\tau$ is

$$\alpha_\Theta^{\vartheta_\tau}(\mathcal{T}_\eta) = \frac{1}{\mathrm{P}\left[\, \|\Theta\| \in \mathcal{I}_\tau \,\right]} \int_{\mathcal{I}_\tau} \left(1 - \mathcal{R}(\rho, \eta)\right) \mathrm{P}\|\Theta\|^{-1}(\mathrm{d}\rho), \tag{20}$$

if $\mathcal{T}_\eta$ and the signal norm testing problem are both from above and

$$\alpha_\Theta^{\vartheta_\tau}(\mathcal{T}_\eta) = \frac{1}{\mathrm{P}\left[\, \|\Theta\| \in \mathcal{I}_\tau \,\right]} \int_{\mathcal{I}_\tau^c} \mathcal{R}(\rho, \eta)\, \mathrm{P}\|\Theta\|^{-1}(\mathrm{d}\rho), \tag{21}$$

if $\mathcal{T}_\eta$ and the signal norm testing problem are both from below.

(iii) The size of $\mathcal{T}_\eta$ for testing $\left[\, \|\Theta\| \in \mathcal{I}_\tau \,\right]$ with $\Theta \in \vartheta_\tau$ is $\alpha^{\vartheta_\tau}(\mathcal{T}) = 1 - \mathcal{R}(\tau, \eta)$ (resp. $\alpha^{\vartheta_\tau}(\mathcal{T}) = \mathcal{R}(\tau, \eta)$) if $\mathcal{T}_\eta$ and the signal norm testing problem are both from above (resp. from below).

(iv) The power of $\mathcal{T}_\eta$ for testing the norm of a given $\Theta \in \vartheta_\tau$ with respect to $\mathcal{I}_\tau$ is

$$\beta_\Theta^{\vartheta_\tau}(\mathcal{T}_\eta) = \frac{1}{\mathrm{P}\left[\, \|\Theta\| \in \mathcal{I}_\tau^c \,\right]} \int_{\mathcal{I}_\tau^c} \left(1 - \mathcal{R}(\rho, \eta)\right) \mathrm{P}\|\Theta\|^{-1}(\mathrm{d}\rho) \tag{22}$$

if $\mathcal{T}_\eta$ and the signal norm testing problem are both from above and

$$\beta_\Theta^{\vartheta_\tau}(\mathcal{T}_\eta) = \frac{1}{\mathrm{P}\left[\, \|\Theta\| \in \mathcal{I}_\tau^c \,\right]} \int_{\mathcal{I}_\tau^c} \mathcal{R}(\rho, \eta)\, \mathrm{P}\|\Theta\|^{-1}(\mathrm{d}\rho), \tag{23}$$

if $\mathcal{T}_\eta$ and the signal norm testing problem are both from below.

PROOF:

Proof of statement **(i)**: Since $\mathcal{T}_\eta$ is a thresholding test, $\mathcal{T}_\eta$ has SIP*fun*. Therefore, according to proposition 5, for any $\Theta \in \vartheta_\tau$, there exists some support $\mathfrak{D}$ of $\mathrm{P}\|\Theta\|^{-1}$ such that $\mathcal{T}_\eta$ has Θ-ISCP*fun* over any sphere with radius in $\mathfrak{D}$. Therefore, given any $\rho \in \mathfrak{D}$, $\beta_\Theta\left(\mathcal{T}_\eta \,\middle|\, \|\Theta\| = \rho\right) = \beta_\theta(\mathcal{T}_\eta)$ for any $\theta \in \rho\mathrm{S}^{d-1}$. From (1) and (4), we derive that $\beta_\Theta\left(\mathcal{T}_\eta \,\middle|\, \|\Theta\| = \rho\right) = 1 - \mathcal{R}(\rho,\eta)$ if the thresholding is from above and that $\beta_\Theta\left(\mathcal{T}_\eta \,\middle|\, \|\Theta\| = \rho\right) = \mathcal{R}(\rho,\eta)$ is the thresholding is from below. Each of the foregoing equalities is equivalent to (19) by definition of the SCP*fun* of test $\mathcal{T}_\eta$.

Proof of statement **(ii)**: A straightforward application of statement **(i)** above and (15).

Proof of statement **(iii)**: Suppose that $\mathcal{T}_\eta$ is from above (resp. from below). From statement **(ii)** above, (11) and the increasingness (resp. decreasingness) of $1-\mathcal{R}(\cdot,\rho)$ (resp. $\mathcal{R}(\cdot,\rho)$) guaranteed by lemma 1, we derive that $\alpha^{\vartheta_\tau}(\mathcal{T}_\eta) \leqslant 1 - \mathcal{R}(\tau,\eta)$ (resp. $\alpha^{\vartheta_\tau}(\mathcal{T}_\eta) \leqslant \mathcal{R}(\tau,\eta)$). To prove that these inequalities are, in fact, equalities, let ρ be any element of $\mathcal{I}_\tau$ and let $\Theta = \varepsilon\theta + (1-\varepsilon)\theta' \in \vartheta_\tau$ where $\theta, \theta' \in \mathbb{R}^d$ with $\|\theta\| = \rho$, $\|\theta'\| = \rho' \in \mathcal{I}_\tau^c$ and ε is Bernouilli distributed, valued in $\{0,1\}$ such that $\mathrm{P}\left[\varepsilon = 1\right] \in (0,1)$. Since $\{\rho,\rho'\}$ is included in every support of $\mathrm{P}\|\Theta\|^{-1}$, (17) (resp. (18)) holds true. We have $\mathrm{P}\|\Theta\|^{-1} = \mathrm{P}\left[\varepsilon = 1\right]\delta_\rho + \mathrm{P}\left[\varepsilon = 0\right]\delta_{\rho'}$ where, given $x \in \mathbb{R}$, δ_x is the Dirac measure centred on x: for any Borel subset A of $\mathbb{R}^d$, $\delta_x(A) = 1$ if $x \in A$ and $\delta_x(A) = 0$, otherwise. It thus follows from (14) that $\mathrm{P}\left[\mathcal{T}_\eta(\Theta + X) = 1 \,\middle|\, \|\Theta\| \in \mathcal{I}_\tau\right]$ equals $1 - \mathcal{R}(\rho,\eta)$ (resp. $\mathcal{R}(\rho,\eta)$). By definition of $\alpha^{\vartheta_\tau}(\mathcal{T}_\eta)$ given by (11), we now have $1 - \mathcal{R}(\rho,\eta) \leqslant \alpha^{\vartheta_\tau}(\mathcal{T}_\eta)$ (resp. $\mathcal{R}(\rho,\eta) \leqslant \alpha^{\vartheta_\tau}(\mathcal{T}_\eta)$). Since ρ is arbitrary in $\mathcal{I}_\tau$, it follows from the continuity of $\mathcal{R}(\rho,\cdot)$ that $\lim_{\rho\to\tau}\mathcal{R}(\rho,\eta) = \mathcal{R}(\tau,\eta)$ so that $1 - \mathcal{R}(\tau,\eta) \leqslant \alpha^{\vartheta_\tau}(\mathcal{T}_\eta)$ (resp. $\mathcal{R}(\tau,\eta) \leqslant \alpha^{\vartheta_\tau}(\mathcal{T}_\eta)$), which concludes the proof of statement **(ii)**.

Proof of statement **(iv)**: A direct application of statement **(i)** and (16). ■

Because the problem of testing $\left[\,\|\Theta\| \in \mathcal{I}_\tau\,\right]$ is spherically invariant, we can expect, for a given $\Theta \in \vartheta_\tau$ and a given ρ in some support $\mathfrak{D}$ of $\mathrm{P}\|\Theta\|^{-1}$, the existence of a test with specified level $\gamma \in (0,1)$ and best Θ-ISCP*fun* over $\rho\mathrm{S}^{d-1}$. By test with level $\gamma \in (0,1)$ and best Θ-ISCP*fun* over $\rho\mathrm{S}^{d-1}$, we mean a test $\mathcal{T} \in \mathcal{K}_\gamma^{\vartheta_\tau}$ such that $\beta_\Theta\left(\mathcal{T} \,\middle|\, \|\Theta\| = \rho\right) \geqslant \beta_\Theta\left(\mathcal{T}' \,\middle|\, \|\Theta\| = \rho\right)$ for any other test $\mathcal{T}' \in \mathcal{K}_\gamma^{\vartheta_\tau}$ with Θ-ISCP*fun* over $\rho\mathrm{S}^{d-1}$. The optimality of such a test would be limited to Θ, whereas our goal is to point out tests that are optimal, in a certain sense related to spherical invariance, for testing $\left[\,\|\Theta\| \in \mathcal{I}_\tau\,\right]$ for all $\Theta \in \vartheta_\tau$. Thence, the following definition.

Definition 6 Given $\gamma \in (0,1)$, a test $\mathcal{T}^*$ is said to have *uniformly best invariant spherically-conditioned power function* — and we say that $\mathcal{T}^*$ is UBISCP — with level (resp. size) γ for testing $\left[\,\|\Theta\| \in \mathcal{I}_\tau\,\right]$ with $\Theta \in \vartheta_\tau$ if:

[Level] : $\mathcal{T}^* \in \mathcal{K}_\gamma^{\vartheta_\tau}$ (resp. $\alpha^{\vartheta_\tau}(\mathcal{T}^*) = \gamma$);

[Power] : for any $\Theta \in \vartheta_\tau$, there exists some support $\mathfrak{D}$ of $\mathrm{P}\|\Theta\|^{-1}$ such that:

[P1] for any $\rho \in \mathfrak{D}$, $\mathcal{T}^*$ has Θ-ISCP*fun* over $\rho\mathrm{S}^{d-1}$,

[P2] for any ρ in $\mathfrak{D} \cap \mathcal{I}_\tau^c$ and any $\mathcal{T} \in \mathcal{K}_\gamma^{\vartheta_\tau}$ with Θ-ISCP*fun* over $\rho\mathrm{S}^{d-1}$,

$$\beta_\Theta\left(\mathcal{T}^* \,\middle|\, \|\Theta\| = \rho\right) \geqslant \beta_\Theta\left(\mathcal{T} \,\middle|\, \|\Theta\| = \rho\right).$$

Remark 2 It follows from statements **(i)** of proposition 5 and lemma 3 that a UBISCP test with level γ for testing $[\, \|\Theta\| \in \mathcal{I}_\tau \,]$ with $\Theta \in \vartheta_\tau$ has necessarily SIP*fun* and level γ for testing $[\, \|\theta\| \in \mathcal{I}_\tau \,]$ with $\theta \in \mathbb{R}^d$.

Remark 3 It is worth emphasizing that property **[P2]** satisfied by UBISCP tests is rather strong. Indeed, if $\mathcal{T}^*$ is UBISCP in $\mathcal{K}_\gamma^{\vartheta_\tau}$, this properties specifies that $\mathcal{T}^*$ has larger Θ-ISCP*fun* over any sphere $\rho \mathrm{S}^{d-1}$ with $\rho \in \mathfrak{D} \cap \mathcal{I}_\tau^c$ than any other test $\mathcal{T}$ with Θ-ISCP*fun* over this same sphere, whatever the behaviour of $\mathcal{T}$ on any sphere other than $\rho \mathrm{S}^{d-1}$. This property induces the following results. In particular, theorem 2 below states that the class of UBISCP tests involves that of UMP-SIP tests.

Proposition 6 Let Θ be some element of ϑ_τ and $\mathfrak{D}$ be some support of $\mathrm{P}\|\Theta\|^{-1}$. Let us consider the class $\mathcal{K}_{\gamma;\Theta\text{-ISCP}fun}^{\vartheta_\tau}$ of those elements of $\mathcal{K}_\gamma^{\vartheta_\tau}$ that have Θ-ISCP*fun* over the spheres with radii in $\mathfrak{D} \cap \mathcal{I}_\tau^c$. If test $\mathcal{T}^*$ is UBISCP with level $\gamma \in (0,1)$ for testing $[\, \|\Theta\| \in \mathcal{I}_\tau \,]$ with $\Theta \in \vartheta_\tau$, then $\mathcal{T}^*$ is UMP within $\mathcal{K}_{\gamma;\Theta\text{-ISCP}fun}^{\vartheta_\tau}$.

PROOF: An application of (16) and definition 6. ■

Theorem 2 If $\mathcal{T}^*$ is UBISCP with level γ for testing $[\, \|\Theta\| \in \mathcal{I}_\tau \,]$ with $\Theta \in \vartheta_\tau$, then $\mathcal{T}^*$ is UMP-SIP with level γ for testing $[\, \|\theta\| \in \mathcal{I}_\tau \,]$ with $\theta \in \mathbb{R}^d$.

PROOF: Let us assume that $\mathcal{T}^*$ is UBISCP with level γ for testing $[\, \|\Theta\| \in \mathcal{I}_\tau \,]$ with $\Theta \in \vartheta_\tau$. As noticed in remark 2, this test is necessarily an element of $\mathcal{K}_{\mathrm{SIP}fun} \cap \mathcal{K}_\gamma$. The only thing to prove is that $\mathcal{T}^*$ is UMP within $\mathcal{K}_{\mathrm{SIP}fun} \cap \mathcal{K}_\gamma$. To this end, let θ be any element of $\mathbb{R}^d$ such that $\|\theta\| \in \mathcal{I}_\tau^c$ and $\mathcal{T}$ be any test in $\mathcal{K}_{\mathrm{SIP}fun} \cap \mathcal{K}_\gamma$. We must show that $\beta_\theta(\mathcal{T}^*) \geqslant \beta_\theta(\mathcal{T})$.

Let us choose some $\theta' \in \mathbb{R}^d$ such that $\theta' \in \mathcal{I}_\tau$ and construct $\Theta = \varepsilon\theta + (1-\varepsilon)\theta' \in \mathcal{I}_\tau$ where ε is Bernouilli distributed, valued in $\{0,1\}$ with $\mathrm{P}[\,\varepsilon = 1\,] \in (0,1)$. We then have $\mathcal{T} \in \mathcal{K}_\gamma^{\vartheta_\tau}$, $\mathcal{T}$ has Θ-ISCP*fun* over $\|\theta\|\mathrm{S}^{d-1}$ and $\beta_\theta(\mathcal{T}) = \beta_\Theta\left(\mathcal{T} \,\middle|\, \|\Theta\| = \|\theta\|\right)$: the first property follows from statement **(ii)** of proposition 5, the second one directly results from statement **(i)** of proposition 5 and the third property is obtained by direct computation or as a straightforward consequence of definition 5. The inequality $\beta_\theta(\mathcal{T}^*) \geqslant \beta_\theta(\mathcal{T})$ then derives from these properties of $\mathcal{T}$ and the fact that $\mathcal{T}^*$ is UBISCP with level γ for testing $[\, \|\Theta\| \in \mathcal{I}_\tau \,]$ with $\Theta \in \vartheta_\tau$ so that $\mathcal{T}^* \in \mathcal{K}_\gamma^{\vartheta_\tau}$, $\mathcal{T}^*$ has Θ-ISCP*fun* over $\|\theta\|\mathrm{S}^{d-1}$, $\beta_\theta(\mathcal{T}^*) = \beta_\Theta\left(\mathcal{T}^* \,\middle|\, \|\Theta\| = \|\theta\|\right) \geqslant \beta_\Theta\left(\mathcal{T} \,\middle|\, \|\Theta\| = \|\theta\|\right)$. ■

The question is now whether UBISCP tests actually exist. The answer is yes, according to our main theorem 3 below. In fact, the previous result implies that UBISCP tests are necessarily UMP-SIP tests. It is thus natural to wonder whether the thresholding tests of theorem 1 are not, in fact, UBISCP for testing the norm of a random signal. Theorem 3 establishes that these tests are indeed UBISCP, which extends their properties stated in theorem 1. Theorem 1 thus turns out to be a direct consequence of theorems 2 and 3.

Theorem 3 Let γ be an element of $(0,1)$. Any thresholding test from above (resp. from below) whose threshold height is $\lambda_\gamma(\tau)$ (resp. $\lambda_{1-\gamma}(\tau)$) is UBISCP with size γ and unbiased for testing $[\, \|\Theta\| \in \mathcal{I}_\tau \,]$ with $\Theta \in \vartheta_\tau$ and $\mathcal{I}_\tau = [0,\tau]\!]$ (resp. $\mathcal{I}_\tau = [0,\tau]\!]^c$).

PROOF: See appendix IV. ■

Remark 4 According to this theorem and the discussion of section 4.2, Wald's test with size γ is UBISCP with size γ for testing $\Theta = 0$ (a-s) with $\Theta \in \vartheta_\tau$.

6 UBISCP tests in signal detection

The detection of an unknown and non-null d-dimensional signal in independent AWGN is a problem of most interest in practice. In many papers and textbooks, the unknown signal is considered to be deterministic. Depending on the geometrical structure that this deterministic unknown signal may satisfy — for instance, if this signal obeys a linear subspace model —, the natural spherical- and scale-invariances of the detection problem can be taken into account so as to reduce the problem via the invariance principle [1–3]. Tests proposed in [4–8] and other works cited in the aforementioned papers are then optimal within a restricted class of tests invariant to nuisance parameters, among which the noise standard deviation. Such tests often relate to the generalized likelihood ratio test (GLRT) for the natural invariance this test can exhibit by involving maximum likelihood estimates of nuisance parameters [5–8]. The so-called subspace adaptive detectors also derive from this invariance principle applied to situations where the noise matrix covariance is unknown and auxiliary data are available [9–19].

For the same type of reasons as those described in the introduction, an unknown but random model for the signal might be prefered in practice. In this respect, the present section addresses the problem of detecting a random signal with unknown distribution in independent AWGN. With no additional assumption, this problem is cast in the SNT framework and the contribution brought by UBISCP tests to signal detection is discussed. We begin by addressing the case of a known noise standard deviation. In subsection 6.2, we consider the case where this standard deviation is unknown and the detection is performed via an estimate-and-plug-in detector based on a noise reference.

6.1 Detection of a random signal

Let Ξ be some d-dimensional real random signal whose distribution is unknown and such that $\Xi \neq 0$ (a-s). As usual, we assume that Ξ is independent with AWGN. As above, noise will be denoted by X. In this subsection, the noise standard deviation is assumed to be known and, without loss of generality, equal to 1. We thus have $X \sim \mathcal{N}(0, \mathbf{I}_d)$. An appropriate framework for the description of detection problems of this type in signal processing is that of binary hypothesis testing (see [27–30]). The so-called null hypothesis $\mathcal{H}_0$ is that only noise is present and the alternative hypothesis $\mathcal{H}_1$ is that the observation is the sum of signal and noise. We always can assume the existence of some non-negative real value τ_0, possibly equal to the trivial lower bound 0 for the norm, such that $\|\Xi\| > \tau_0$ (a-s). Denoting the observation by Y, the problem of detecting Ξ in noise X can then be summarized by

$$\begin{cases} \mathcal{H}_0 : Y \sim \mathcal{N}(0, \mathbf{I}_d), \\ \mathcal{H}_1 : Y = \Xi + X, X \sim \mathcal{N}(0, \mathbf{I}_d), \mathrm{P}\left[\, \|\Xi\| > \tau_0 \,\right] = 1. \end{cases} \tag{24}$$

The performance of a given test $\mathcal{T}$, that is, a measurable map of $\mathbb{R}^d$ into $\{0, 1\}$, is then measured via the false alarm and detection probabilities. The false alarm probability is the probability of erroneously accepting the null hypothesis $\mathcal{H}_1$ when the observation is noise only, that is, the probability $\mathrm{P}_{\mathrm{FA}}\left[\mathcal{T}\right] = \mathrm{P}\left[\,\mathcal{T}(X) = 1\,\right]$. The detection probability is the probability of correctly accepting the alternative hypothesis $\mathcal{H}_1$, that is, the probability $\mathrm{P}_{\mathrm{D}}\left[\mathcal{T}\right] = \mathrm{P}\left[\,\mathcal{T}(\Xi + X) = 1\,\right]$. The detection problem (24) can then be cast in the theoretical SNT framework of section 5.

To see this, we first assume the existence of a random variable ε independent of Ξ and X, defined on the same probability space as Ξ and X, valued in $\{0,1\}$ and such that $Y = \varepsilon\Xi + X$. The signal is present (resp. absent) whenever $\varepsilon = 1$ (resp. $\varepsilon = 0$). Given any test $\mathcal{T}$, the value of the random variable $\mathcal{T}(Y) = \mathcal{T} \circ Y$ is the index of the accepted hypothesis, whereas the value of ε is the index of the true hypothesis. With a slight and easy extension of the terminology introduced in section 5, we could also say that detecting the presence or the absence of Ξ in independent AWGN amounts to testing the event $\left[\,\varepsilon = 0\,\right]$ against the event $\left[\,\varepsilon = 1\,\right]$. The introduction of the indicator variable ε induces that of the signal prior probabilities of presence $\mathrm{P}\left[\,\varepsilon = 1\,\right]$ and absence $\mathrm{P}\left[\,\varepsilon = 0\,\right]$, in contrast to the standard Neymann-pearson approach, wich avoids this. However, the role of these priors is very limited and merely convenient to state and treat the problem within the SNT theoretical framework of section 5. For the problem to be meaningful, we assume that $\mathrm{P}\left[\,\varepsilon = 1\,\right] \in (0,1)$.

Now, set $\Theta = \varepsilon\Xi$. For any non-negative real value τ such that $0 \leqslant \tau \leqslant \tau_0$, the events $\left[\,\|\Theta\| \leqslant \tau\,\right]$ and $\left[\,\|\Theta\| > \tau\,\right]$ are P – (a-s) equal to the events $\left[\,\varepsilon = 0\,\right]$ and $\left[\,\varepsilon = 1\,\right]$, respectively. Thereby, $\mathrm{P}\left[\,\|\Theta\| \leqslant \tau\,\right] = \mathrm{P}\left[\,\varepsilon = 0\,\right]$ and $\mathrm{P}\left[\,\|\Theta\| > \tau\,\right] = \mathrm{P}\left[\,\varepsilon = 1\,\right]$ so that Θ is an element of ϑ_τ since $\mathrm{P}\left[\,\varepsilon = 0\,\right]$ and $\mathrm{P}\left[\,\varepsilon = 1\,\right]$ are both elements of $(0,1)$. Consequently, the detection problem (24) is the SNT problem of testing the event $\left[\,\|\Theta\| \leqslant \tau\,\right]$, up to a negligible P – negligible subset of Ω. In other words, making a decision about the presence or the absence of Ξ in independent AWGN amounts to testing $\left[\,\|\Theta\| \in \mathcal{I}_\tau\,\right]$ with $\Theta \in \vartheta_\tau$, $\mathcal{I}_\tau = [0,\tau]$ and $\tau \in [0,\tau_0]$. This is SNT from above with respect to $\mathcal{I}_\tau$. According to theorem 3, there exists a UBISCP test with level $\gamma \in (0,1)$ for this SNT problem. This UBISCP test is the thresholding test from above $\mathcal{T}_{\lambda_\gamma(\tau)}$ with threshold height $\lambda_\gamma(\tau)$.

With regard to what follows, we now calculate the false alarm and detection probabilities of any given thresholding test from above $\mathcal{T}_\eta$ with threshold height η. The false alarm probability of $\mathcal{T}_\eta$ is

$$\mathrm{P_{FA}}\left[\,\mathcal{T}_\eta\,\right] = \mathrm{P}\left[\,\|X\| > \eta\,\right] = 1 - \mathcal{R}(0,\eta) \leqslant 1 - \mathcal{R}(\tau,\eta), \tag{25}$$

which follows from (1) and the increasingness of $1 - \mathcal{R}(\cdot,\eta)$ induced by lemma 1. Since the detection problem (24) is an SNT problem, we can easily verify that (25) actually derives from results of section 5. In fact, thanks to the independence of ε, Ξ and X, we have $\mathrm{P_{FA}}\left[\,\mathcal{T}_\eta\,\right] = \alpha_\Theta^{\vartheta_\tau}(\mathcal{T}_\eta) \leqslant \alpha^{\vartheta_\tau}(\mathcal{T}_\eta)$. Since $\mathrm{P}\|\Theta\|^{-1} = \mathrm{P}\left[\,\varepsilon = 0\,\right]\delta_0 + \mathrm{P}\left[\,\varepsilon = 1\,\right]\mathrm{P}\|\Xi\|^{-1}$ and $\|\Xi\| > \tau$ (a-s), (25) results from (20) and statement **(iii)** of lemma 4. As far as the detection probability of $\mathcal{T}_\eta$ is concerned, we have $\mathrm{P_D}\left[\,\mathcal{T}_\eta\,\right] = \beta_\Theta^{\vartheta_\tau}(\mathcal{T}_\eta)$, because of the independence of ε, Ξ and X. By taking the expression of $\mathrm{P}\|\Theta\|^{-1}$ given above and the fact that $\|\Xi\| > \tau_0$ (a-s), we derive from (22) and the decreasingness of $\mathcal{R}(\cdot,\eta)$ guaranteed by lemma 1 that

$$\mathrm{P_D}\left[\,\mathcal{T}_\eta\,\right] = \int_{\tau_0}^{\infty} (1 - \mathcal{R}(\rho,\eta))\,\mathrm{P}\|\Xi\|^{-1}(\mathrm{d}\rho) \geqslant 1 - \mathcal{R}(\tau_0,\eta). \tag{26}$$

By applying (25) and taking the definition of $\lambda_\gamma(\tau)$ into account, we derive that the false alarm probability of the UBISCP test $\mathcal{T}_{\lambda_\gamma(\tau)}$ satisfies

$$\mathrm{P_{FA}}\left[\,\mathcal{T}_{\lambda_\gamma(\tau)}\,\right] = 1 - \mathcal{R}(0,\lambda_\gamma(\tau)) \leqslant \gamma. \tag{27}$$

In the same way, it results from (26) that the detection probability of $\mathcal{T}_{\lambda_\gamma(\tau)}$ is lower bounded by

$$\mathrm{P_D}\left[\,\mathcal{T}_{\lambda_\gamma(\tau)}\,\right] \geqslant 1 - \mathcal{R}(\tau_0,\lambda_\gamma(\tau)). \tag{28}$$

It is usual to characterize the performance of a family of tests with levels in $(0,1)$ by the receiver operator characteristic (ROC) curve of this family of tests. Each point M of the ROC curve of this family is obtained for a given level γ, the abscissa of M being the false alarm probability of the test with level γ and the ordinate of M being the detection probability of this same test. For a given tolerance τ, we can thus consider the ROC curve of the family of UBISCP tests $\{T_{\lambda_\gamma(\tau)} : \gamma \in (0,1)\}$. This ROC curve is the set of points $\mathcal{C}[\mathcal{T}_{\lambda_\gamma(\tau)}] = \left\{(\mathrm{P_{FA}}[\mathcal{T}_{\lambda_\gamma(\tau)}], \mathrm{P_D}[\mathcal{T}_{\lambda_\gamma(\tau)}]) : \gamma \in (0,1)\right\}$. We can also consider the set of points

$$\widehat{\mathcal{C}}[\mathcal{T}_{\lambda_\gamma(\tau)}] = \left\{\left(1-\mathcal{R}(0,\lambda_\gamma(\tau)), 1-\mathcal{R}(\tau_0,\lambda_\gamma(\tau))\right) : \gamma \in (0,1)\right\}. \tag{29}$$

This curve is hereafter called the *lower ROC curve* since, according to (27) and (28), it lies below the ROC one. When τ ranges in $[0,\tau_0]$, the families of UBISCP tests $\{T_{\lambda_\gamma(\tau)} : \gamma \in (0,1)\}$ have all the same ROC and the same lower ROC curves. This simply follows from the fact that, given $\mathcal{T}_{\lambda_\gamma(\tau)}$ with $\tau \in [0,\tau_0]$ and any $\tau' \in [0,\tau_0]$, statement **(iii)** of lemma 2 guarantees the existence of a unique $\gamma' \in (0,1)$ such that $\lambda_\gamma(\tau) = \lambda_{\gamma'}(\tau')$. The difference in performance between the UBISCP tests $\mathcal{T}_{\lambda_\gamma(\tau)}$ when τ ranges in $[0,\tau_0]$ can then be exhibited by observing, for a given level $\gamma \in (0,1)$, the false alarm probability and the lower bound for the detection probability when τ varies. In fact, when the tolerance τ increases to τ_0, the false alarm probability and the lower bound for the detection probability of $\mathcal{T}_{\lambda_\gamma(\tau)}$ both decrease and the former tends from above to $1-\mathcal{R}(0,\lambda_\gamma(\tau_0))$, whereas the latter tends from above to $1-\mathcal{R}(\tau_0,\lambda_\gamma(\tau_0))$. In contrast, when the tolerance τ decreases to 0, the false alarm probability and the lower bound for the detection probability of $\mathcal{T}_{\lambda_\gamma(\tau)}$ both increase, the former tending from below to the specified level, whereas the latter tends from below to $1-\mathcal{R}(\tau_0,\lambda_\gamma(0))$. This behaviour straighforwardly derives from the properties of $\mathcal{R}$ and λ_γ and is coherent with the fact that the UBISCP tests have same ROC curve. As an illustration of this discussion, figure 1 displays the false alarm probabilities and the lower bounds for the detection probabilities of several UBISCP tests $\mathcal{T}_{\lambda_\gamma(\tau)}$ when $d = 12$ and the signal norm lower bound is $\tau_0 = 7$.

According to the foregoing, we can conclude this section by saying that, unless the application requires a false alarm probability actually lesser than the specified level γ, the most appropriate UBISCP test for detecting the signal is Wald's test with size γ, which is discussed in section 4.2 and remark 4. Indeed, the detection probability lower bound yielded by this test is the largest possible one, whereas the false alarm of this test remains equal to the specified level. However, there exist situations where the flexibility on the actual size of the UBISCP tests proves helpful. We describe such a situation in the next section.

6.2 Detection in noise with unknown standard deviation

We now consider the case where the noise standard deviation is unknown but auxiliary data are available to consitute a noise reference. The following discussion emphasizes that an estimate-and-plug-in detector based on UBISCP tests can be used to cope with such a situation and brings some robustness. An estimate-and-plug-in detector basically involves estimating the noise standard deviation on the basis of the noise reference and using this estimate instead of the true value in the expression of a test designed for the nominal case of a known standard deviation [27, Chapter 9, p. 337].

We begin with an easy remark. Suppose that the unknown value of the noise standard deviation is 1 again and that a measurement of this value, say σ, has been provided by some device, once for all. Let us consider that this measurement is deterministic. When Wald's test with size γ is adjusted with σ, we obtain the thresholding test $\mathcal{T}_{\sigma\lambda_\gamma(0)}$ with threshold height $\sigma\lambda_\gamma(0)$. The false alarm probability of this test is $\mathrm{P_{FA}}\left[\mathcal{T}_{\sigma\lambda_\gamma(0)}\right] = \mathrm{P}\left[\|X\| > \sigma\lambda_\gamma(0)\right] = 1 - \mathcal{R}(0, \sigma\lambda_\gamma(0))$. If $\sigma < 1$, the strict increasingness of $\mathcal{R}(0,\cdot)$ implies that this false alarm probability is lower bounded by $1 - \mathcal{R}(0, \lambda_\gamma(0))$, which equals γ, by definition of λ_γ. Therefore, when a noise standard deviation measurement less than 1 is used to adjust the estimate-and-plug-in detector based on Wald's test with size γ, the resulting test has a false alarm probability above the specified level γ, which is undesirable. It follows that the conclusion of the previous section may fail in practical cases where an estimate of the noise standard deviation is plugged into the expression of the test. The use of a UBISCP test with non-null tolerance can therefore be expected to avoid this unwanted behaviour because, as emphasized in the previous section, a non-null tolerance lowers the size of the UBISCP test for detecting the signal.

Instead of further detailling the example above, let us tackle the more general situation where the estimate-and-plug-in detector is adjusted with some estimate $\widehat{\sigma}$ of the noise standard deviation. We assume that $\widehat{\sigma}$, X and Ξ are independent. Without loss of generality because of the scale invariance of the problem, let us assume that the noise standard deviation is 1 again. The thresholding test with threshold height $\lambda_\gamma(\tau)$ with $\tau \in [0, \tau_0]$ is UBISCP for testing $\left[\|\Theta\| \in \mathcal{I}_\tau\right]$ with $\Theta \in \vartheta_\tau$ and $\mathcal{I}_\tau = [0, \tau]$. By replacing the actual value of the noise standard deviation by its estimate $\widehat{\sigma}$ in the expression of this UBISCP test, we do not obtain a test in the sense given above but a UBISCP estimate-and-plug-in detector — in short, UBISCP detector —, which is henceforth denoted by $\mathcal{T}_{\widehat{\sigma}\lambda_\gamma(\tau/\widehat{\sigma})}(Y)$. The UBISCP detector decides that the signal is present if $\|Y\| > \widehat{\sigma}\lambda_\gamma(\tau/\widehat{\sigma})$ and that the signal is absent, otherwise. Once again, the handling of equality in this decision does not matter for the absolute continuity of $\|Y\|$ with respect to Lebesgue's measure in $\mathbb{R}$. The index of the hypothesis accepted by $\mathcal{T}_{\widehat{\sigma}\lambda_\gamma(\tau/\widehat{\sigma})}(Y)$ is thus the value of $\mathcal{T}_{\widehat{\sigma}\lambda_\gamma(\tau/\widehat{\sigma})}(Y)$. The false alarm probability of $\mathcal{T}_{\widehat{\sigma}\lambda_\gamma(\widehat{\sigma})}(Y)$ is then $\mathrm{P_{FA}}\left[\mathcal{T}_{\widehat{\sigma}\lambda_\gamma(\tau/\widehat{\sigma})}(Y)\right] = \mathrm{P}\left[\|X\| > \widehat{\sigma}\lambda_\gamma(\tau/\widehat{\sigma})\right]$. Because of the independence of $\widehat{\sigma}$ and X, we have:

$$\mathrm{P_{FA}}\left[\mathcal{T}_{\widehat{\sigma}\lambda_\gamma(\tau/\widehat{\sigma})}(Y)\right] = \int_0^\infty \mathrm{P_{FA}}\left[\mathcal{T}_{\sigma\lambda_\gamma(\tau/\sigma)}\right]\mathrm{P}\widehat{\sigma}^{-1}(\mathrm{d}\sigma), \tag{30}$$

where $\mathrm{P}\widehat{\sigma}^{-1}$ stands for the probability distribution of $\widehat{\sigma}$ and the false alarm probability $\mathrm{P_{FA}}\left[\mathcal{T}_{\sigma\lambda_\gamma(\tau/\sigma)}\right]$ of the thresholding test $\mathcal{T}_{\sigma\lambda_\gamma(\tau/\sigma)}$ with threshold height $\sigma\lambda_\gamma(\tau/\sigma)$ can be computed according to (25). Similarly, the detection probability of $\mathcal{T}_{\widehat{\sigma}\lambda_\gamma(\widehat{\sigma})}(Y)$ is given by

$$\mathrm{P_D}\left[\mathcal{T}_{\widehat{\sigma}\lambda_\gamma(\tau/\widehat{\sigma})}(Y)\right] = \int_0^\infty \mathrm{P_D}\left[\mathcal{T}_{\sigma\lambda_\gamma(\tau/\sigma)}\right]\mathrm{P}\widehat{\sigma}^{-1}(\mathrm{d}\sigma), \tag{31}$$

where the detection probability $\mathrm{P_D}\left[\mathcal{T}_{\sigma\lambda_\gamma(\tau/\sigma)}\right]$ of the thresholding test $\mathcal{T}_{\sigma\lambda_\gamma(\tau/\sigma)}$ with threshold height $\sigma\lambda_\gamma(\tau/\sigma)$ can be calculated via (26). The detection probability of the UBISCP detector can be lower bounded by applying (26), so that:

$$\mathrm{P_D}\left[\mathcal{T}_{\widehat{\sigma}\lambda_\gamma(\tau/\widehat{\sigma})}(Y)\right] \geqslant \int_0^\infty (1 - \mathcal{R}(\tau_0, \sigma\lambda_\gamma(\tau/\sigma))\mathrm{P}\widehat{\sigma}^{-1}(\mathrm{d}\sigma). \tag{32}$$

As in section 6.1, the lower ROC curve of the UBISCP detector $\mathcal{T}_{\widehat{\sigma}\lambda_\gamma(\tau/\widehat{\sigma})}(Y)$ is defined as the set of points

$$\widehat{\mathcal{C}}[\mathcal{T}_{\widehat{\sigma}\lambda_\gamma(\tau/\widehat{\sigma})}(Y)] = \left\{ \left(\mathrm{P_{FA}}\left[\mathcal{T}_{\widehat{\sigma}\lambda_\gamma(\tau/\widehat{\sigma})}(Y) \right], \int_0^\infty (1 - \mathcal{R}(\tau_0, \sigma\lambda_\gamma(\tau/\sigma)) \mathrm{P}\widehat{\sigma}^{-1}(\mathrm{d}\sigma) \right) : \gamma \in (0,1) \right\}.$$

On the one hand, the larger the rhs in (32), the larger the detection probability of $\mathcal{T}_{\widehat{\sigma}\lambda_\gamma(\tau/\widehat{\sigma})}(Y)$. The largest possible value for the rhs in (32) is

$$\int_0^\infty (1 - \mathcal{R}(\tau_0, \sigma\lambda_\gamma(0/\sigma))\, \mathrm{P}\widehat{\sigma}^{-1}(\mathrm{d}\sigma).$$

This value is the detection probability lower bound of the Wald estimate-and-plug-in detector — in short, Wald detector — $\mathcal{T}_{\widehat{\sigma}\lambda_\gamma(0/\widehat{\sigma})}(Y)$, which derives from Wald's test $\mathcal{T}_{\lambda_\gamma(0)}$ by replacing the known unitary standard deviation by $\widehat{\sigma}$. Therefore, a suitable tolerance τ should be as small as possible so as to guarantee a detection probability lower bound close to that of the Wald detector. However, a too small value for τ may not be appropriate for the following reason. According to the properties of $\mathcal{R}$ and λ_γ stated in section 3, $\mathrm{P_{FA}}\left[\mathcal{T}_{\widehat{\sigma}\lambda_\gamma(\tau/\widehat{\sigma})}(Y)\right]$ is a continuous and decreasing function of τ and thus, since $\tau \in [0, \tau_0]$, we have $\mathrm{P_{FA}}\left[\mathcal{T}_{\widehat{\sigma}\lambda_\gamma(\tau_0/\widehat{\sigma})}(Y)\right] \leqslant \mathrm{P_{FA}}\left[\mathcal{T}_{\widehat{\sigma}\lambda_\gamma(\tau/\widehat{\sigma})}(Y)\right] \leqslant \mathrm{P_{FA}}\left[\mathcal{T}_{\widehat{\sigma}\lambda_\gamma(0)}(Y)\right]$. The upper bound in this inequality is the false alarm probability of the Wald detector $\mathcal{T}_{\widehat{\sigma}\lambda_\gamma(0)}(Y)$. If the false alarm probability $\mathrm{P_{FA}}\left[\mathcal{T}_{\widehat{\sigma}\lambda_\gamma(0)}(Y)\right]$ of the Wald detector is above γ, τ should therefore not be chosen too close to 0. It follows from the above remarks that the UBISCP detector for a given level γ should be the *adjusted*-UBISCP (A-UBISCP) detector $\mathcal{T}_{\widehat{\sigma}\lambda_\gamma(\tau^*/\widehat{\sigma})}(Y)$ with adjusted tolerance $\tau^* = \underset{\tau\in[0,\tau_0]}{\operatorname{argmin}} \left| \mathrm{P_{FA}}\left[\mathcal{T}_{\widehat{\sigma}\lambda_\gamma(\tau/\widehat{\sigma})}(Y)\right] - \gamma \right|$. Thanks to the continuity and strict decreasingness of $\mathrm{P_{FA}}\left[\mathcal{T}_{\widehat{\sigma}\lambda_\gamma(\tau/\widehat{\sigma})}(Y)\right]$ with τ, we have: $\tau^* = \tau_0$ if $\mathrm{P_{FA}}\left[\mathcal{T}_{\widehat{\sigma}\lambda_\gamma(\tau_0/\widehat{\sigma})}(Y)\right] \geqslant \gamma$; $\tau^* = 0$ if $\mathrm{P_{FA}}\left[\mathcal{T}_{\widehat{\sigma}\lambda_\gamma(0)}(Y)\right] \leqslant \gamma$; τ^* is the unique solution in τ to the equation $\mathrm{P_{FA}}\left[\mathcal{T}_{\widehat{\sigma}\lambda_\gamma(\tau/\widehat{\sigma})}(Y)\right] = \gamma$ if $\mathrm{P_{FA}}\left[\mathcal{T}_{\widehat{\sigma}\lambda_\gamma(\tau_0/\widehat{\sigma})}(Y)\right] < \gamma < \mathrm{P_{FA}}\left[\mathcal{T}_{\widehat{\sigma}\lambda_\gamma(0)}(Y)\right]$. In the last case, the adjusted tolerance τ^* guarantees a false alarm probability of the A-UBISCP detector $\mathcal{T}_{\widehat{\sigma}\lambda_\gamma(\tau^*/\widehat{\sigma})}(Y)$ equal to the specified level γ and we also have

$$\tau^* = \min\left\{ \tau \in [0, \tau_0] : \mathrm{P_{FA}}\left[\mathcal{T}_{\widehat{\sigma}\lambda_\gamma(\tau/\widehat{\sigma})}(Y)\right] \leqslant \gamma \right\}.$$

If the estimate $\widehat{\sigma}$ is good enough, it can be further expected that the detection performance of the A-UBISCP detector will remain comparable to that achieved when the noise standard deviation is known and the detection is performed by Wald's test.

To prolongate the discussion, let us consider the case where $\widehat{\sigma}$ is the noise standard deviation maximum likelihood estimate (MLE) calculated on the basis of a noise reference. More specifically, suppose we are given an N-dimensional random vector $W \sim \mathcal{N}(0, \mathbf{I}_N)$, independent of X and Ξ. This vector is a noise reference and the estimate $\widehat{\sigma}$ is now the noise standard deviation MLE

$$\widehat{\sigma}_N = \frac{1}{\sqrt{N}} \|W\|. \tag{33}$$

Since $\|W\|^2$ follows the centred chi-2 distribution with N degrees of freedom, the probability distribution $\mathrm{P}\widehat{\sigma}_N^{-1}$ of $\widehat{\sigma}_N$ has density $f_N(\sigma) = 2N\sigma f_{\chi^2_N(0)}(N\sigma^2)$, where $f_{\chi^2_N(0)}$ stands for the probability density function of the centred chi-2 distribution with N degrees of freedom. The A-UBISCP detector is then the MLE A-UBISCP detector $\mathcal{T}_{\widehat{\sigma}_N\lambda_\gamma(\tau_N^*/\widehat{\sigma}_N)}(Y)$ where τ_N^* is the adjusted tolerance τ^* calculated with $\widehat{\sigma} = \widehat{\sigma}_N$. In this case, we can make the following remarks.

The MLE $\widehat{\sigma}_N$ is strongly consistent so that $\widehat{\sigma}_N$ tends to 1 (a-s). Therefore,

$$\lim_{N\to\infty} \mathrm{P_{FA}}\left[\,\mathcal{T}_{\widehat{\sigma}_N \lambda_\gamma(\tau_0/\widehat{\sigma}_N)}(Y)\,\right] = \mathrm{P_{FA}}\left[\,\mathcal{T}_{\lambda_\gamma(\tau_0)}\,\right] = 1 - \mathcal{R}(0, \lambda_\gamma(\tau_0)).$$

It then follows from the strict increasingness of $1-\mathcal{R}(\cdot,\lambda_\gamma(\tau_0))$ guaranteed by lemma 1 and the definition of $\lambda_\gamma(\tau_0)$ given by lemma 2 that $1-\mathcal{R}(0,\lambda_\gamma(\tau_0)) < \gamma$, so that $\mathrm{P_{FA}}\left[\,\mathcal{T}_{\widehat{\sigma}_N \lambda_\gamma(\tau_0/\widehat{\sigma}_N)}(Y)\,\right] < \gamma$ for N large enough. Therefore, for N above some natural integer, τ_N^* is either 0 or such that $\mathrm{P_{FA}}\left[\,\mathcal{T}_{\widehat{\sigma}_N \lambda_\gamma(\tau_N^*/\widehat{\sigma}_N)}(Y)\,\right] = \gamma$. Since the MLE strong consistency also implies that $\lim_{N\to\infty} \mathrm{P_{FA}}\left[\,\mathcal{T}_{\widehat{\sigma}_N \lambda_\gamma(0)}(Y)\,\right] = \mathrm{P_{FA}}\left[\,\mathcal{T}_{\lambda_\gamma(0)}\,\right] = 1-\mathcal{R}(0,\lambda_\gamma(0)) = \gamma$, we can conjecture that τ_N^* tends to 0. The actual proving of this conjecture is still an open issue. However, we can establish the existence of a subsequence of $\{\tau_N^* : N = 1, 2, \ldots\}$ that converges to 0. Indeed, suppose the existence of some positive real value τ such that $\tau_N^* \geqslant \tau$ for any large enough integer N. On the one hand, for any integer N large enough, we would have $\mathrm{P_{FA}}\left[\,\mathcal{T}_{\widehat{\sigma}_N \lambda_\gamma(\tau_N^*/\widehat{\sigma}_N)}(Y)\,\right] = \gamma$. On the other hand, for N large enough again, it would follow from (30) and the decreasingness of $1-\mathcal{R}(0,\cdot)$ that

$$\mathrm{P_{FA}}\left[\,\mathcal{T}_{\widehat{\sigma}_N \lambda_\gamma(\tau_N^*/\widehat{\sigma}_N)}(Y)\,\right] \leqslant \int_0^\infty (1-\mathcal{R}(0,\sigma\lambda_\gamma(\tau/\sigma)))\, \mathrm{P}\widehat{\sigma}_N^{-1}(\mathrm{d}\sigma).$$

Since the rhs in the inequality above tends to $1-\mathcal{R}(0,\lambda_\gamma(\tau))$ when N tends to ∞ because of the MLE strong consistency, we would necessary have $\gamma \leqslant 1-\mathcal{R}(0,\lambda_\gamma(\tau))$, a contradiction since $\tau > 0$, $1-\mathcal{R}(\cdot,\lambda_\gamma(\tau))$ is strictly increasing and $\gamma = 1-\mathcal{R}(\tau,\lambda_\gamma(\tau))$ by lemmas 1 and 2. For a subsequence of tolerances that converges to 0, the lower ROC curve of the resulting MLE A-UBISCP detectors will thus approach that obtained when the noise standard deviation is known and the detection is performed by Wald's test.

The foregoing discussion is illustrated by figures 2 and 3. On the one hand, figure 2 shows that the false alarm probability of the MLE Wald detector — that is, the Wald detector adjusted with the MLE of (33) — is above the specified level, whereas the MLE A-UBISCP detector guarantees the specified level. On the other hand, figure 3 presents the lower ROC curve of the MLE A-UBISCP detector for comparison to that obtained, when the noise standard deviation is known, by Wald's test. The false alarm probabilities and the detection probability lower bounds displayed in these figures were numerically calculated by standard quadrature Gaussian integration based on the expression of the probability distribution of $\widehat{\sigma}_N$ and `MATLAB` routines of the toolbox `stats`, since $\mathcal{R}(\rho,\eta) = F_{\chi_d^2(\rho^2)}(\eta^2)$ where $F_{\chi_d^2(\rho^2)}$ is the cumulative distribution function of the non-central χ^2 distribution with d degrees of freedom and non-central parameter ρ^2.

7 Conclusion and perspectives

In this paper, we have introduced the SNT problem in presence of independent AWGN and proposed a theoretical framework dedicated to this type of problem. Basically, the problem is to decide whether some random signal, whose distribution is unknown and which is observed in independent AWGN with known variance, has norm above or below some tolerance that can be specified by the user himself, on the basis of his own experience and know-how with respect to a given environment or context. The theoretical framework proposed in this paper has led to several results. In particular, we have established the existence of UBISCP tests, which are optimal with respect to a suitable spherical invariance-based criterion for the SNT problem. The theoretical results established in this paper encompass standard ones obtained when the signal is unknown deterministic and, in fact, make it possible to embrace a whole class of testing problems within a unified theoretical framework. We can also say, as in [31, Sec. 3.1, p. 1160] about Wald's UBCP tests, that the UBISCP tests are alternative to tests, such as likelihood ratio tests, whose power is optimal for a certain class of signals but that can be very inefficient over the complementary of this class. As illustrated in section 6 dedicated to the detection of any random signal with any unknown distribution, the use of a positive tolerance in SNT and the properties of the UBISCP tests bring some robustness, when the noise standard deviation is unknown and the detection is performed via an estimate-and-plug-in detector. The application of the SNT framework to the detection of random signals with unknown distributions in independent AWGN should be further studied in combination with results stated in the reference papers mentioned in section 6. Besides, a complete study of the MLE A-UBISCP, involving the case of a non signal-free reference, could impact the design of constant false alarm rate (CFAR) systems standardly used in radar processing [32, 33] and, lately, in ultra wideband (UWB) receivers [34]. In this respect, for the estimation of the noise standard deviation on the basis of a non signal-free reference, it would also be desirable to analyse to what extent SNT could be combined to standard results in robust statistics [35–38], as well as to [39, 40] that propose robust noise standard deviation estimates in presence of any random signals obeying sparsity hypotheses. Beyond the standard detection problem, SNT and UBISCP tests should also apply to many other practical problems, among which those evoked in the introduction, as soon as the problem is the detection of a deviation from a nominal reference.

We now emphasize some theoretical prospects opened by this paper. Our discussion relies on the fact that the actual main crux in the aproach is the invariance of the noise probability distribution. The event to test regarding the signal thus involves the norm because the norm is readily the most straightforward maximal invariant of the orthogonal group in $\mathbb{R}^d$. Therefore, the noise geometrical properties — in terms of invariance with respect to the orthogonal group — have induced the type of events to test in SNT, as well as the type of tests to use in SNT, whatever the signal distribution. We could also say that the noise properties are sufficient to exhibit a large class of event testing problems that can be solved without prior knowledge on the signal distributions. It is thus rather natural to wonder to what extent such an approach, constrained by the noise invariance only, can actually be extended so as to deal with other types of noise. For instance, let us consider some noise X whose distribution is invariant with respect to a certain group $\mathcal{G}$. The question is whether, similarly to the SNT problem addressed in this paper, event testing problems could be specified and solved via extended notions of SIP*fun*, tests with Θ-ISCP*fun* and UBISCP tests, defined on the basis of some maximal invariant ν of $\mathcal{G}$. It is thinkable that the answer to such a question should strongly depends on whether the distribution of $\nu(X)$ has a monotone likelihood ratio and whether a function similar to $\mathcal{R}$ defined by (1) actually exists.

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Appendix I
Proof of lemma 1

This improvement of [41, Lemma IV.2] is proved similarly by refining some arguments. Let ρ and ρ' be two real numbers such that $0 \leqslant \rho < \rho' < \infty$. Let θ and θ' be two colinear vectors of $\mathbb{R}^d$ such that $\|\theta\| = \rho$ and $\|\theta'\| = \rho'$. According to (1), $\mathcal{R}(\rho,\eta) = \int_{B(\theta,\eta)} f(x)\,\mathrm{d}x$ and $\mathcal{R}(\rho',\eta) = \int_{B(\theta',\eta)} f(x)\,\mathrm{d}x$ where f is the probability density function of X and $B(\theta,\eta)$ (resp. $B(\theta',\eta)$) is the closed ball, in $\mathbb{R}^d$, centred at θ (resp. θ') with radius η. We have $\mathcal{R}(\rho,\eta) - \mathcal{R}(\rho',\eta) = \int_{B(\theta,\eta)\setminus B(\theta',\eta)} \big(f(x) - f(\theta+\theta'-x)\big)\,\mathrm{d}x$. Let $(e_1, e_2, \ldots, e_d)$ be an orthonormal basis of $\mathbb{R}^d$ such that $\theta = \rho e_1$ and $\theta' = \rho' e_1$. We have $\|\theta+\theta'-x\|^2 - \|x\|^2 = (\rho+\rho')(\rho+\rho'-2x_1)$ for any $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$. If $x \in B(\theta,\eta)\setminus B(\theta',\eta)$, then $\|x-\theta'\| > \|x-\theta\|$, which implies that $(\rho'-\rho)(\rho+\rho'-2x_1) > 0$ and, thus, that $\rho+\rho'-2x_1 > 0$ since $\rho' > \rho$. Therefore, $\|\theta+\theta'-x\| > \|x\|$. Since f decreases strictly with the norm of its argument, it follows that $f(x) - f(\theta+\theta'-x) > 0$ so that $\mathcal{R}(\rho,\eta) > \mathcal{R}(\rho',\eta)$ and the proof is complete.

Appendix II
Proof of lemma 2

[Existence and unicity of $\lambda_\gamma(\rho)$] : $\mathcal{R}(\rho,\cdot)$ is a one-to-one mapping from $[0,\infty)$ into $[0,1)$. Thence, the existence and the unicity of $\lambda_\gamma(\rho)$ for $\gamma \in (0,1]$.

[Strict increasingness of λ_γ] : Let ρ and ρ' be two non-negative real number such that $\rho < \rho'$. According to lemma 1, $\mathcal{R}(\rho',\lambda_\gamma(\rho)) < \mathcal{R}(\rho,\lambda_\gamma(\rho))$. The right hand side (rhs) in this inequality equals $1-\gamma$ and, thus, $\mathcal{R}(\rho',\lambda_\gamma(\rho'))$. The result then follows from the strict increasingness of $\mathcal{R}(\rho',\cdot)$.

[Continuity of λ_γ] : Given $\rho_0 \in [0,\infty)$, the strict increasingness of λ_γ implies the existence of a limit $\lambda_\gamma(\rho_0^-) \in [0,\infty)$ when ρ tends to ρ_0 from below and the existence of a limit $\lambda_\gamma(\rho_0^+) \in [0,\infty)$ when ρ tends to ρ_0 from above. Since $\mathcal{R}$ is continuous in the plane and $\mathcal{R}(\rho,\lambda_\gamma(\rho)) = 1-\gamma$ for every $\rho \in [0,\infty)$, $\mathcal{R}(\rho_0,\lambda_\gamma(\rho_0^-)) = \mathcal{R}(\rho_0,\lambda_\gamma(\rho_0^+)) = 1-\gamma$. Since $\mathcal{R}(\rho_0,\cdot)$ is one-to-one, $\lambda_\gamma(\rho_0^-) = \lambda_\gamma(\rho_0^+) = \lambda_\gamma(\rho_0)$ and λ_γ is continuous.

[Strict decreasingness of $\gamma \mapsto \lambda_\gamma(\rho)$] : Let ρ be some element of $[0,\infty)$. let us consider two elements γ and γ' of $(0,1]$. We have $1-\mathcal{R}(\rho,\lambda_\gamma(\rho)) = \gamma$ and $1-\mathcal{R}(\rho,\lambda_{\gamma'}(\rho)) = \gamma'$. If $\gamma < \gamma'$, we thus have $\mathcal{R}(\rho,\lambda_\gamma(\rho)) > \mathcal{R}(\rho,\lambda_{\gamma'}(\rho))$, which implies that $\lambda_\gamma(\rho) > \lambda_{\gamma'}(\rho)$ since $\mathcal{R}(\rho,\cdot)$ is strictly increasing.

[Continuity of $\gamma \mapsto \lambda_\gamma(\rho)$] : The proof is similar to that of the continuity of λ_γ and left to the reader.

Appendix III
Proof of proposition 5

III.1 Proof of statement (i)

We begin with the direct implication. Given any $\rho \in [0,\infty)$, let $\rho\mathrm{S}^{d-1}$ stand for the sphere with radius ρ and centred at the origin in $\mathbb{R}^d$. If $\mathcal{T}$ has SIP*fun*, we can define the map $\mathfrak{R}$ of $[0,\infty)$ into $[0,1]$ such that $\mathfrak{R}(\rho) = \mathrm{P}\left[\mathcal{T}(\theta + X) = 1\right]$ for any $\theta \in \rho\mathrm{S}^{d-1}$.

Let Θ be some element of ϑ_τ and B be any Borel set of $\mathbb{R}$. From the independence of Θ and X, it follows from the definition of $\mathcal{R}$ that, for any $\theta \in \mathbb{R}^d$,

$$\mathrm{P}\left[\mathcal{T}(\Theta + X) = 1\,,\, \|\Theta\| \in B \,\middle|\, \Theta = \theta\right] = \mathrm{I}_B(\|\theta\|)\,\mathfrak{R}(\|\theta\|) \tag{34}$$

where I_B is the indicator function of B: $\mathrm{I}_B(x) = 1$ if $x \in B$ and $\mathrm{I}_B(x) = 0$, otherwise. By the standard change-of-variable formula [42, Theorem 16.13], we now have

$$\int \mathrm{I}_B(\|\theta\|)\,\mathfrak{R}(\|\theta\|)\,\mathrm{P}\Theta^{-1}(\mathrm{d}\theta) = \int_B \mathfrak{R}(\rho)\,\mathrm{P}\|\Theta\|^{-1}(\mathrm{d}\rho). \tag{35}$$

Since $\mathrm{P}\left[\mathcal{T}(\Theta + X) = 1\,,\, \|\Theta\| \in B\right] = \int \mathrm{P}\left[\mathcal{T}(\Theta + X) = 1\,,\, \|\Theta\| \in B \,\middle|\, \Theta = \theta\right]\mathrm{P}\Theta^{-1}(\mathrm{d}\theta)$, it follows from (34) and (35) that

$$\mathrm{P}\left[\mathcal{T}(\Theta + X) = 1\,,\, \|\Theta\| \in B\right] = \int_B \mathfrak{R}(\rho)\,\mathrm{P}\|\Theta\|^{-1}(\mathrm{d}\rho).$$

On the other hand,

$$\mathrm{P}\left[\mathcal{T}(\Theta + X) = 1\,,\, \|\Theta\| \in B\right] = \int_B \mathrm{P}\left[\mathcal{T}(\Theta + X) = 1 \,\middle|\, \|\Theta\| = \rho\right]\mathrm{P}\|\Theta\|^{-1}(\mathrm{d}\rho). \tag{36}$$

Therefore, we derive from the foregoing and the definition of a conditional probability that

$$\mathrm{P}\left[\mathcal{T}(\Theta + X) = 1 \,\middle|\, \|\Theta\| = \rho\right] = \mathfrak{R}(\rho),\ \mathrm{P}\|\Theta\|^{-1} - \text{(a-s)}.$$

We now establish the converse statement. Let $\mathcal{T}$ be some test. Assume that, for any $\Theta \in \mathcal{M}(\Omega, \mathbb{R}^d)$, there exists a support $\mathfrak{D}$ of $\mathrm{P}\|\Theta\|^{-1}$ such that $\mathcal{T}$ has Θ-ISCP*fun* over any sphere with radius in $\mathfrak{D}$. Let ρ be any non-negative real number and Θ be any element of $\mathcal{M}(\Omega, \mathbb{R}^d)$ such that $\|\Theta\| = \rho$ (a-s). Since ρ belongs to any support of Θ, $\mathcal{T}$ has Θ-ISCP*fun* over $\rho\mathrm{S}^{d-1}$. We thus have $\beta_\theta(\mathcal{T}) = \beta_{\theta'}(\mathcal{T})$ for any $\theta' \in \rho\mathrm{S}^{d-1}$. Thence, the spherical invariance of the power function of $\mathcal{T}$ since ρ has been chosen arbitrarily.

III.2 Proof of statement (ii)

With the same assumptions as above, let us suppose that $\mathcal{T}$ has level $\gamma \in [0,1]$ for testing $\left[\,\|\theta\| \in \mathcal{I}_\tau\right]$ with $\theta \in \mathbb{R}^d$, so that $\beta_\theta(\mathcal{T}) \leqslant \alpha(\mathcal{T}) \leqslant \gamma$ for every $\theta \in \mathbb{R}^d$ such that $\|\theta\| \in \mathcal{I}_\tau$. It then follows from statement **(i)** proved above that, for every $\Theta \in \vartheta_\tau$, $\beta_\Theta\left(\mathcal{T} \,\middle|\, \|\Theta\| = \rho\right) \leqslant \gamma$ for $\mathrm{P}\|\Theta\|^{-1}$ – almost every $\rho \in \mathcal{I}_\tau$. Statement **(ii)** then follows from (15).

Appendix IV
Proof of theorem 3

We begin with some technical preliminary results. In particular, corollary 1 of lemma 5 is an interesting result beyond the scope of the present paper. In fact, although this corollary is probably standard [5], we did not find precise references. So, we have provided a proof that derives from the following lemma 5. Many arguments given in this section rely on the following analytical expression of $\mathcal{R}$ [41, Section V, p. 232, (19)], which straightforwardly follows from [43, p. 22, Theorem 1.3.4]. For every pair (ρ,η) of non-negative real numbers, we have

$$\mathcal{R}(\rho,\eta) \stackrel{\Delta}{=} \frac{e^{-\rho^2/2}}{2^{d/2-1}\Gamma(d/2)}\int_0^\eta e^{-t^2/2}t^{d-1}{}_0\mathrm{F}_1\big(d/2;\rho^2t^2/4\big)\,\mathrm{d}t \tag{37}$$

where ${}_0\mathrm{F}_1$ is the generalized hypergeometric function [44, p. 275].

Lemma 5 For every real number $\nu \geqslant 1/2$ and every pair of non-negative real numbers ρ_0 and ρ_1 such that $0 \leqslant \rho_0 < \rho_1$, the continuous map

$$x \in [0,\infty) \longmapsto {}_0\mathrm{F}_1\left(\nu;\rho_1^2x^2/4\right)/{}_0\mathrm{F}_1\left(\nu;\rho_0^2x^2/4\right)$$

is strictly increasing and, thus, one-to-one.

PROOF: Set $f(x) = {}_0\mathrm{F}_1\left(\nu;\rho_1^2x^2/4\right)/{}_0\mathrm{F}_1\left(\nu;\rho_0^2x^2/4\right)$ for any $x \in [0,\infty)$. We have $f(0)=1$ and since ${}_0\mathrm{F}_1(\nu;\cdot)$ is increasing, we have $f(x) \geqslant 1$ for any $x \geqslant 0$. For $x \in (0,\infty)$, the derivative of ${}_0\mathrm{F}_1(\nu;x)$ with respect to x follows from [44, Sec. 9.14, p.275] and some routine algebra shows that the sign of $f'(x)$ is that of

$$q(x) = \frac{\rho_1^2}{\rho^2}\frac{{}_0\mathrm{F}_1(\nu+1;\rho_1^2x^2/4)}{{}_0\mathrm{F}_1(\nu;\rho_1^2x^2/4)} - \frac{{}_0\mathrm{F}_1(\nu+1;\rho_0^2x^2/4)}{{}_0\mathrm{F}_1(\nu;\rho_0^2x2/4)}.$$

Put $g(t) = I_\nu(t)/I_{\nu-1}(t)$, $t \in [0,\infty)$, where I_ν is the modified Bessel function [45, Sec. 9.6, p. 374]. According to [45, p. 377, 9.6.47], we have $g(t) = \frac{t}{2\nu}\frac{{}_0\mathrm{F}_1(\nu+1;t^2/4)}{{}_0\mathrm{F}_1(\nu;t^2/4)}$, $t \in [0,\infty)$. Therefore, $q(x) = \frac{2\nu}{\rho_0^2x}\left(\rho_1 g(\rho_1 x) - \rho_0 g(\rho_0 x)\right)$ whose sign is that of $\rho_1 g(\rho_1 x) - \rho_0 g(\rho_0 x)$. It is proved in [41, Lemma B.1, Appendix B, p. 237] that g is strictly increasing. Therefore, since $\rho_0 < \rho_1$ and g is non-negative, we have $\rho_0 g(\rho_0 x) < \rho_1 g(\rho_1 x)$ and the proof is complete. ∎

Corollary 1 *The family of the non-central χ^2 distributions with d degrees of liberty has monotone likelihood ratio with its non-central parameter.*

Proposition 7 Let $\gamma \in (0,1]$. Given any two non-negative real values ρ_0 and ρ_1 such that $\rho_0 < \rho_1$, let Ξ_0 and Ξ_1 be any random vectors that are uniformly distributed on $\rho_0\mathrm{S}^{d-1}$ and $\rho_1\mathrm{S}^{d-1}$, respectively. Any thresholding test from above with threshold height $\lambda_\gamma(\rho_0)$ is most powerful (MP) with size γ for testing the null hypothesis $\mathcal{H}_0 : Y = \Xi_0 + X$ against the alternative one $\mathcal{H}_1 : Y = \Xi_1 + X$. The power of this MP test is $1 - \mathcal{R}(\rho_1, \lambda_\gamma(\rho_0))$.

PROOF: It follows from [41, Proposition V.1, (18), p. 232] that the likelihood ratio for testing $\mathcal{H}_0$ against $\mathcal{H}_1$ is

$$\Lambda(y) = \exp\left(-(\rho_1^2-\rho_0^2)/2\right){}_0\mathrm{F}_1\left(d/2;\rho_1^2\|y\|^2/4\right)/{}_0\mathrm{F}_1\left(d/2;\rho_0^2\|y\|^2/4\right), y \in \mathbb{R}^d.$$

According to the Neyman-Pearson lemma [1, Theorem 3.2.1, Sec. 3.2, p. 60], there exists some constant ζ with the following property: each test that accepts (resp. rejects) $\mathcal{H}_0$ if $\Lambda(y) < \zeta$ (resp. $\Lambda(y) > \zeta$) is MP with size γ for testing $\mathcal{H}_0$ againts $\mathcal{H}_1$. According to lemma 5, this MP test is a thresholding test from above $\mathcal{T}_{\zeta'}$. The threshold height ζ' of this test can be calculated by solving the equation $\mathrm{P}\left[\mathcal{T}_{\zeta'}(\Xi_0+X)=1\right]=\gamma$. Since we derive from (1) and lemma 4 — or, equivalently, from (37) and [41, Proposition V.1, (17), p. 232] — that $\mathrm{P}\left[\mathcal{T}_{\zeta'}(\Xi_0+X)=1\right]=1-\mathcal{R}(\rho_0,\zeta')$, it follows that $\zeta'=\lambda_\gamma(\rho_0)$. The power of the thresholding test from above with threshold height $\lambda_\gamma(\rho_0)$ for testing $\mathcal{H}_0$ against $\mathcal{H}_1$ is a consequence of (1) and lemma 4 — or (37) and [41, Proposition V.1, (17), p. 232] — again. ■

We now tackle the proof of theorem 3 in the case where SNT is from above tolerance τ, that is, when we test the event $\left[\,\|\Theta\|\leqslant\tau\,\right]$ with $\Theta\in\vartheta_\tau$ and $\mathcal{I}_\tau=[0,\tau⟦$. Indeed, the proving when the testing is from below τ, that is, when $\mathcal{I}_\tau=⟧\tau,\infty)$, can be carried out by mimicking what follows and is left to the reader.

We thus consider any thresholding test from above $\mathcal{T}_{\lambda^*}$, whose threshold height is $\lambda^*=\lambda_\gamma(\tau)$. The fact that $\alpha^{\vartheta_\tau}(\mathcal{T}_{\lambda^*})=\gamma$ is a direct consequence of lemma 4, statement **(ii)**. Let Θ be any element of ϑ_τ. According to statement **(i)** of lemma 4, there exists some support $\mathfrak{D}$ of $\mathrm{P}\|\Theta\|^{-1}$ such that $\mathcal{T}_{\lambda^*}$ has Θ-ISCP*fun* over any sphere with radius in $\mathfrak{D}$. It thus remains to prove that $\mathcal{T}_{\lambda^*}$ satisfies property **[P2]** of definition 6 and that $\mathcal{T}_{\lambda^*}$ is unbiased. To this end, let $\mathcal{T}$ be some element of $\mathcal{K}_\gamma^{\vartheta_\tau}$ such that $\mathcal{T}$ has Θ-ISCP*fun* over any sphere with radius in $\mathfrak{D}\cap\mathcal{I}_\tau^c$. Given any $\rho_0\in\mathcal{I}_\tau$ and any $\rho_1\in\mathfrak{D}\cap\mathcal{I}_\tau^c$, let Ξ_0 and Ξ_1 be any two elements of $\mathcal{M}(\Omega,\mathbb{R}^d)$ independent of X such that $\|\Xi_0\|=\rho_0$ (a-s) and $\|\Xi_1\|=\rho_1$ (a-s). Let ε be some random variable valued in $\{0,1\}$, independent of Ξ_0, Ξ_1 and X, with $\mathrm{P}\left[\varepsilon=1\right]\in(0,1)$. We have $\mathrm{P}\left[\mathcal{T}(\Xi_0+X)=1\right]=\mathrm{P}\left[\mathcal{T}(\Xi+X)=1\,\middle|\,\|\Xi\|\in\mathcal{I}_\tau\right]$, where $\Xi=(1-\varepsilon)\Xi_0+\varepsilon\Xi_1\in\mathcal{I}_\tau$. Thereby, since $\mathcal{T}\in\mathcal{K}_\gamma^{\vartheta_\tau}$, we obtain that:

$$\mathrm{P}\left[\mathcal{T}(\Xi_0+X)=1\right]\leqslant\gamma. \tag{38}$$

Now, because $\mathcal{T}$ is assumed to have Θ-ISCP*fun* over $\rho_1\mathrm{S}^{d-1}$, we derive from proposition 4 that

$$\mathrm{P}\left[\mathcal{T}(\Xi_1+X)=1\right]=\beta_\Theta\left(\mathcal{T}\,\middle|\,\|\Theta\|=\rho_1\right). \tag{39}$$

Let us consider the problem of testing the null hypothesis $\mathcal{H}_0: Y=\Xi_0+X$ against the alternative one $\mathcal{H}_1: Y=\Xi_1+X$. It then follows from (38) and (39) that $\mathcal{T}$ has level γ and power equal to $\beta_\Theta\left(\mathcal{T}\,\middle|\,\|\Theta\|=\rho\right)$ for testing $\mathcal{H}_0$ against $\mathcal{H}_1$. This holds true for any distribution of Ξ_0 and Ξ_1, provided that the supports of these random vectors are $\rho_0\mathrm{S}^{d-1}$ and $\rho_1\mathrm{S}^{d-1}$, respectively. We then choose Ξ_0 (resp. Ξ_1) uniformly distributed over $\rho_0\mathrm{S}^{d-1}$ (resp. $\rho_1\mathrm{S}^{d-1}$). According to proposition 7, the thresholding test from above $\mathcal{T}_{\lambda_\gamma(\rho_0)}$ with threshold height $\lambda_\gamma(\rho_0)$ is MP with size γ and power equal to $1-\mathcal{R}(\rho_1,\lambda_\gamma(\rho_0))$ for testing $\mathcal{H}_0$ against $\mathcal{H}_1$. As a consequence, we have $1-\mathcal{R}(\rho_1,\lambda_\gamma(\rho_0))\geqslant\beta_\Theta\left(\mathcal{T}\,\middle|\,\|\Theta\|=\rho_1\right)$. The left hand side in the previous inequality tends to $1-\mathcal{R}(\rho_1,\lambda^*)$ when ρ_0 tends to τ by continuity of $\mathcal{R}(\rho_1,\cdot)$ and λ_γ (see lemma 2). Since $\mathcal{T}_{\lambda^*}$ has Θ-ISCP*fun* over $\rho_1\mathrm{S}^{d-1}$, (17) induces that $1-\mathcal{R}(\rho_1,\lambda^*)=\beta_\Theta\left(\mathcal{T}^*\,\middle|\,\|\Theta\|=\rho_1\right)$. Therefore, $\beta_\Theta\left(\mathcal{T}^*\,\middle|\,\|\Theta\|=\rho_1\right)\geqslant\beta_\Theta\left(\mathcal{T}\,\middle|\,\|\Theta\|=\rho_1\right)$.

We now show that $\mathcal{T}_{\lambda^*}$ is unbiased. According to (22), we have

$$\beta_\Theta^{\vartheta_\tau}(\mathcal{T}_{\lambda^*})=\frac{1}{\mathrm{P}\left[\,\|\Theta\|\in\mathcal{I}_\tau^c\right]}\int_{\mathcal{I}_\tau^c}\left(1-\mathcal{R}(\rho,\lambda^*)\right)\mathrm{P}\|\Theta\|^{-1}(\mathrm{d}\rho)$$

with $\mathcal{I}_\tau = [0, \tau]\![$, since we consider SNT from above. We also have $1 - \mathcal{R}(\rho, \lambda^*) \geqslant 1 - \mathcal{R}(\tau, \lambda^*)$ for $\rho \in \mathcal{I}_\tau^c$ thanks to lemma 1. Thence, the unbiasedness of $\mathcal{T}_{\lambda^*}$ because $1 - \mathcal{R}(\tau, \lambda^*) = \gamma$.

References

[1] E. L. Lehmann and J. P. Romano, *Testing Statistical Hypotheses, Third edition.* Springer, 2005.

[2] A. A. Borovkov, *Mathematical statistics.* Gordon and Breach Science Publishers, 1998.

[3] M. L. Eaton, *Multivariate statistics. A vector space approach.* Wiley, 1983.

[4] L. L. Scharf and D. Lytle, "Signal detection in gaussian noise of unknown level: An invariance application," *IEEE Transactions on Information Theory*, vol. 17, no. 4, pp. 404–411, July 1971.

[5] L. L. Scharf and B. Friedlander, "Matched subspace detectors," *IEEE Transactions on Signal Processing*, vol. 42, no. 8, pp. 2146–2157, 1994.

[6] S. M. Kay and J. R. Gabriel, "Optimal invariant detection of a sinusoid with unknown parameters," *IEEE Transactions on Signal Processing*, vol. 50, no. 1, pp. 27–40, January 2002.

[7] S. Kay and J. Gabriel, "An invariance property of the generalized likelihood ratio test," *IEEE Transactions on Signal Processing*, vol. 10, no. 12, pp. 352–355, December 2003.

[8] J. Gabriel and S. Kay, "On the relationship between the GLRT and UMPI tests for the detection of signals with unknown parameters," *IEEE Transactions on Signal Processing*, vol. 53, no. 11, pp. 4193–4203, November 2005.

[9] E. Kelly, "An adaptive detection algorithm," *Aerospace and Electronic Systems, IEEE Transactions on*, vol. AES-22, no. 2, pp. 115 –127, March 1986.

[10] S. Bose and A. Steinhardt, "A maximal invariant framework for adaptive detection with structured and unstructured covariance matrices," *IEEE Transactions on Signal Processing*, vol. 43, no. 9, pp. 2164–2175, September 1995.

[11] E. Conte, M. Lops, and G. Ricci, "Asymptotically optimum radar detection in compound-Gaussian clutter," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 31, no. 2, pp. 617–625, April 1995.

[12] ——, "Adaptive matched filter detection in spherically invariant noise," *IEEE Signal Processing Letters*, vol. 3, no. 8, pp. 248–250, August 1996.

[13] S. Kraut and L. L. Scharf, "The CFAR adaptive subspace detector is a scale-invariant GLRT," *IEEE Transactions on Signal Processing*, vol. 47, no. 9, pp. 2538 –2541, September 1999.

[14] S. Kraut, L. L. Scharf, and L. T. McWhorter, "Adaptive subspace detectors," *IEEE Transactions on Signal Processing*, vol. 49, no. 1, pp. 1–16, January 2001.

[15] E. Conte, A. De Maio, and G. Ricci, "Recursive estimation of the covariance matrix of a compound-gaussian process and its application to adaptive CFAR detection," *IEEE Transactions on Signal Processing*, vol. 50, no. 8, pp. 1908 – 1915, August 2002.

[16] E. Conte, A. De Maio, and C. Galdi, "CFAR detection of multidimensional signals: an invariant approach," *IEEE Transactions on Signal Processing*, vol. 51, no. 1, pp. 142–151, January 2003.

[17] L. L. Scharf, S. Kraut, and M. L. McCloud, "A review of matched and adaptive subspace detectors," in *Adaptive Systems for Signal Processing, Communications, and Control Symposium (AS-SPCC)*, October 2000, pp. 82–86.

[18] S. Kraut, L. L. Scharf, and R. W. Butler, "The adaptive coherence estimator: a uniformly most-powerful-invariant adaptive detection statistic," *IEEE Transactions on Signal Processing*, vol. 53, no. 2, pp. 427 – 438, February 2005.

[19] A. De Maio and E. Conte, "Adaptive detection in gaussian interference with unknown covariance after reduction by invariance," *IEEE Transactions on Signal Processing*, vol. 58, no. 6, pp. 2925–2934, June 2010.

[20] V. Krishnamurthy, "Optimal threshold policies for multivariate stopping-time POMDPs," in *Symbolic and Quantitative Approaches to Reasoning with Uncertainty*. Lecture Notes in Computer Science, Springer, 2009, vol. 5590, pp. 850–862.

[21] I. Miller, M. Campbell, and D. Huttenlocher, "Efficient unbiased tracking of multiple dynamic obstacles under large viewpoint changes," *IEEE Transactions on Robotics*, vol. 27, no. 1, pp. 29 –46, February 2011.

[22] H. Sohn, C. R. Farrar, F. M. Hemez, D. D. Shunk, D. W. Stinemates, B. R. Nadler, and J. J. Czarnecki, "A review of structural health monitoring literature: 1996 – 2001," Los Alamos National Laboratory Report, LA-13976-MS, Tech. Rep., February 2004.

[23] B. Thumati and S. Jagannathan, "A model-based fault-detection and prediction scheme for nonlinear multivariable discrete-time systems with asymptotic stability guarantees," *IEEE Transactions on Neural Networks*, vol. 21, no. 3, pp. 404 –423, March 2010.

[24] D. Gorinevsky, S.-J. Kim, S. Beard, S. Boyd, and G. Gordon, "Optimal estimation of deterioration from diagnostic image sequence," *IEEE Transactions on Signal Processing*, vol. 57, no. 3, pp. 1030 –1043, March 2009.

[25] M. Djeziri, R. Merzouki, and B. Bouamama, "Robust monitoring of an electric vehicle with structured and unstructured uncertainties," *IEEE Transactions on Vehicular Technology*, vol. 58, no. 9, pp. 4710 –4719, November 2009.

[26] A. Wald, "Tests of statistical hypotheses concerning several parameters when the number of observations is large," *Transactions of the American Society*, vol. 54, no. 3, pp. 426–482, November 1943.

[27] S. M. Kay, *Fundamentals of Statistical Signal Processing, Volume II, Detection Theory, 14th printing.* Prenctice Hall, 2009.

[28] T. Kailath and H. V. Poor, "Detection of stochastic processes," *IEEE Transactions on Information Theory*, vol. 44, pp. 2230–2259, 1998.

[29] H. V. Poor, *An Introduction to Signal Detection and Estimation. 2nd Edition.* Springer-Verlag, New York, 1994.

[30] H. V. Trees, *Detection, Estimation, and Modulation Theory - Part l - Detection, Estimation, and Linear Modulation Theory.* John Wiley & Sons, 2001.

[31] M. Fouladirad and I. Nikiforov, "Optimal statistical fault detection with nuisance parameters," *Automatica*, vol. 41, pp. 1157–1171, 2005.

[32] N. Levanon and E. Mozeson, *Radar signals.* Wiley, 2004.

[33] G. Minkler and J. Minkler, *The Principles of Automatic Radar Detection In Clutter,* CFAR. Magellan Book Company, Baltimore, 1990.

[34] A. Maali, A. Mesloub, M. Djeddou, H. Mimoun, G. Baudoin, and A. Ouldali, "Adaptive ca-cfar threshold for non-coherent ir-uwb energy detector receivers," *IEEE Communications letters*, vol. 13, no. 12, pp. 959 – 961, December 2009.

[35] F. Hampel, "The influence curve and its role in robust estimation," *Journal of the American Statistical Association*, vol. 69, no. 346, pp. 383–393, June 1974.

[36] P. Rousseeuw and C. Croux, "Explicit scale estimators with high breakdown point," *L1-Statistical Analysis and Related Methods*, pp. 77–92, 1992.

[37] ——, "Alternatives to the median absolute deviation," *Journal of the American Statistical Association*, vol. 88, no. 424, pp. 1273–1283, December 1993.

[38] P. Huber and E. Ronchetti, *Robust Statistics, second edition.* John Wiley and Sons, 2009.

[39] F.-X. Socheleau, D. Pastor, and A. Aissa-El-Bey, "Robust statistics based noise variance estimation: Application to wideband interception of noncooperative communications," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 47, no. 1, pp. 746 –755, March 2011.

[40] D. Pastor and F.-X. Socheleau, "Algorithms based on sparsity hypotheses for robust estimation of the noise standard deviation in presence of signals with unknown distributions and occurrences, version # 2, RR-2010003-SC," Institut Télécom, Télécom Bretagne, Lab-STICC UMR CNRS 3192, Tech. Rep., 2011.

[41] D. Pastor, R. Gay, and A. Gronenboom, "A sharp upper bound for the probability of error of likelihood ratio test for detecting signals in white gaussian noise," *IEEE Transactions on Information Theory*, vol. 48, no. 1, pp. 228–238, January 2002.

[42] P. Billingsley, *Probability and Measure, third edition.* Wiley, 1995.

[43] R. J. Muirhead, *Aspects of multivariate statistical theory.* Wiley, 1982.

[44] N. Lebedev, *Special Functions and their Applications.* Prentice-Hall, Englewood Cliffs, 1965.

[45] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions with formulas, graphs and mathematical tables.* Dover, 1964.

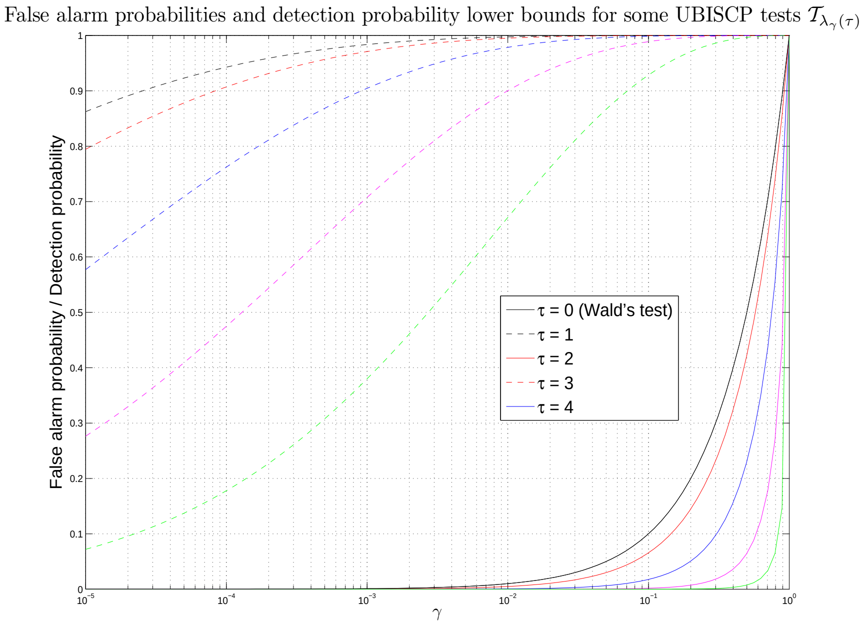

Figure 1: False alarm probabilities and lower bounds for the detection probabilities of UBISCP tests $\mathcal{T}_{\lambda_\gamma(\tau)}$ with $\tau \in \{1,2,3,4\}$, when the signal and noise have dimension $d = 12$ and the lower bound for the signal norm is $\tau_0 = 7$. The abscissas are given in the logarithmic (base 10) scale so that the curves presented in this figure are plotted as functions of $\log_{10}(\gamma)$, with $\gamma \in (0,1)$. The false alarm probability and the detection probability lower bound both decrease (resp. increase) when τ increases (resp. decreases). The dashed curves are the probability detection lower bounds and the solid curves are the false alarm probabilities. The false alarm probability of Wald's test is exactly the curve $\{(\log_{10}(\gamma), \gamma) : \gamma \in (0,1)\}$, below which the false alarm probabilities of all the other UBISCP tests remain. The probability detection lower bound of Wald's test is above those of all the other UBISCP tests. Therefore, when the noise standard deviation is known, Wald's test is the best UBISCP test to use for detecting random signals with unknown distributions.

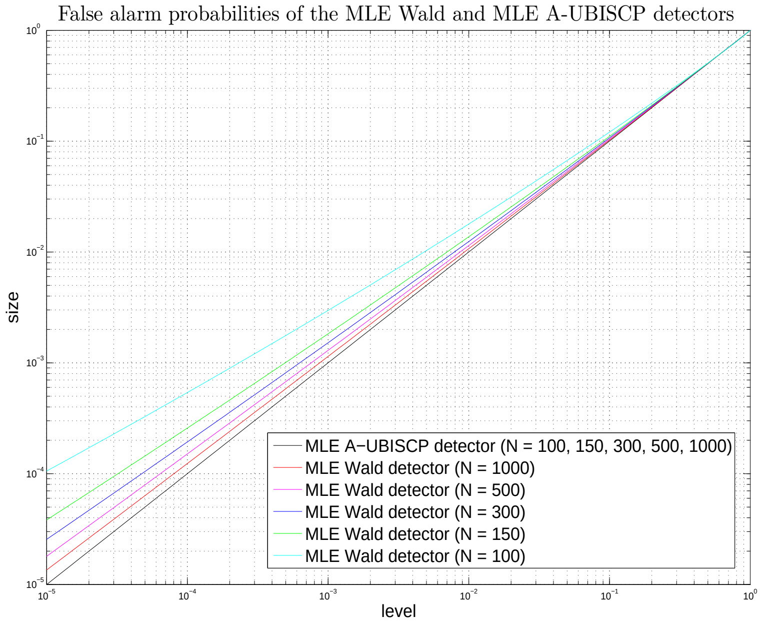

Figure 2: The signal and noise are assumed to have dimension $d = 12$ and the signal norm lower bound is supposed to be $\tau_0 = 7$. This figure displays the false alarm probability of the MLE Wald detector as a function of $\log_{10}(\gamma)$, when $N = 100, 150, 300, 500$ in (33) and the specified level γ ranges in $(0, 1)$. The false alarm probabilities of the MLE Wald detector are above the diagonal and, thus, above the specified level γ, which is undesirable. In contrast, the MLE A-UBISCP detector guarantees the specifed level.

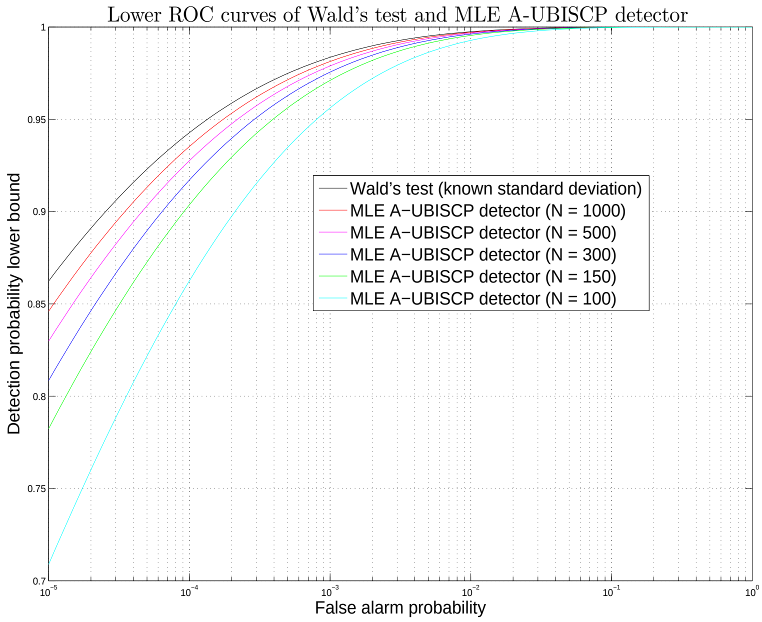

Figure 3: As in figure 2, the signal and noise are assumed to have dimension $d = 12$ and the signal norm lower bound is supposed to be $\tau_0 = 7$. The figure displays the lower ROC curves of the MLE A-UBISCP detector with adjusted tolerance τ_N^*, with $N = 100, 150, 300, 500$ in (33). Whatever the tested value of N, $\mathrm{P_{FA}}\left[\mathcal{T}_{\widehat{\sigma}_N \lambda_\gamma(\tau_0/\widehat{\sigma}_N)}(Y)\right] < \gamma < \mathrm{P_{FA}}\left[\mathcal{T}_{\widehat{\sigma}_N \lambda_\gamma(0)}(Y)\right]$ so that τ_N^* guarantees that the false alarm probability of the MLE A-UBISCP detector equals the specifed level. The use of an estimate of the noise standard deviation impacts the detection performance. Indeed, the lower ROC curves of the MLE A-UBISCP detector are below the lower ROC curve of Wald's test obtained when the noise standard deviation is known. However, this performance loss in detection reduces as N increases, whereas the false alarm probability of the MLE A-UBISCP detector remains equal to the specified level.

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